Jin Akiyama
Edy Tri Baskoro
Mikio Kano (Eds.)

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Volume Editors

Jin Akiyama
Tokai University, Research Institute of Educational Development
Tokyo, 151-8677, Japan
E-mail: fwjb5117@infoweb.ne.jp

Edy Tri Baskoro
Institut Teknologi Bandung, Department of Mathematics
Jalan Ganesa 10, Bandung 40132, Indonesia
E-mail: ebaskoro@dns.math.itb.ac.id

Mikio Kano
Ibaraki University, Department of Computer and Information Sciences
Hitachi, Ibaraki, 316-8511, Japan
E-mail: kano@cis.ibaraki.ac.jp

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Preface

This volume consists of the refereed papers presented at the Indonesia-Japan Joint Conference on Combinatorial Geometry and Graph Theory (IJCCGGT 2003), held on September 13–16, 2003 at ITB, Bandung, Indonesia. This conference can also be considered as a series of the Japan Conference on Discrete and Computational Geometry (JCDCG), which has been held annually since 1997. The first five conferences of the series were held in Tokyo, Japan, the sixth in Manila, the Philippines, in 2001, and the seventh in Tokyo, Japan in 2002.

The proceedings of JCDCG 1998, JCDCG 2000 and JCDCG 2002 were published by Springer as part of the series Lecture Notes in Computer Science: LNCS volumes 1763, 2098 and 2866, respectively. The proceedings of JCDCG 2001 were also published by Springer as a special issue of the journal *Graphs and Combinatorics*, Vol. 18, No. 4, 2002.

The organizers are grateful to the Department of Mathematics, Institut Teknologi Bandung (ITB) and Tokai University for sponsoring the conference. We also thank all program committee members and referees for their excellent work. Our big thanks to the principal speakers: Hajo Broersma, Mikio Kano, Janos Pach and Jorge Urrutia. Finally, our thanks also goes to all our colleagues who worked hard to make the conference enjoyable and successful.

August 2004

Jin Akiyama
Edy Tri Baskoro
Mikio Kano

Organization

The Indonesia-Japan Joint Conference on Combinatorial Geometry and Graph Theory (IJCCGGT) 2003 was organized by the Department of Mathematics, Institut Teknologi Bandung (ITB) Indonesia and RIED, Tokai University, Japan.

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Table of Contents

On Convex Developments of a Doubly-Covered Square

Jin Akiyama[1] and Koichi Hirata[2]

[1] Research Institute of Educational Development,
Tokai University, Tokyo 151-8677, Japan
fwjb5117@mb.infoweb.ne.jp
[2] Faculty of Education, Ehime University,
Matsuyama 790-8577, Japan
hirata@ed.ehime-u.ac.jp

Abstract. We give an algebraic characterization of all convex polygons that are 2-flat foldable to a square, that is, we determine all shapes of convex developments of a doubly-covered square.

1 Doubly-Covered Square

Let us introduce an equivalence relation on the plane $E = \{\,(x,\, y)\,|\, x, y \in \boldsymbol{R}\}$ in the following way.

Definition 1. *We say that two points $X_1(x_1,\, y_1)$ and $X_2(x_2,\, y_2)$ of the plane E are equivalent if either one of the following two conditions is satisfied:*

(1) The points X_1 and X_2 are symmetric with respect to some lattice point. Namely, the midpoint of the line segment X_1X_2 is a lattice point.
(2) The point X_1 can be moved to the point X_2 by means of a parallel translation given by a vector whose components are both even integers. Namely, the components of the vector $\overrightarrow{X_1X_2}$ are both even integers.

The fact that the conditions above define an equivalence relation is obvious since the composition of any combinations of motions, moving points to those which are symmetric in the sense of (1) or parallel translations of the type (2), yields again the motion of the type (1) or (2).

Lemma 1. *Denote by P the quotient space obtained from E by means of the equivalence relation introduced in Definition 1, and denote by p the quotient map $E \to P$. Then, P can be identified with a doubly-covered square (Fig. 1).*

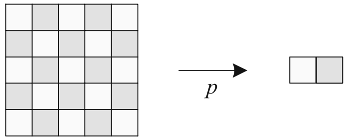

Fig. 1

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Proof. Let us show first that a representative of an equivalence class for the given equivalence relation can be chosen in the set

$$R = \{ (x, y) \mid -1 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Letters k and l represent integers in the sequel.

(a) Any point (x, y) satisfying the conditions $2k-1 \leq x < 2k+1$, $2l-1 \leq y < 2l$ is equivalent to a point in R by means of a motion of the type (1) moving points in E to those which are symmetric with respect to the point (k, l).
(b) Any point (x, y) satisfying the conditions $2k-1 \leq x < 2k+1$, $2l \leq y < 2l+1$ is equivalent to a point in R by means of the parallel translation of the type (2) given by the vector $(-2k, -2l)$.

Therefore, we see that a representative of an equivalence class can be chosen in the set R. We next investigate equivalence of the points within R by means of the given equivalence relation. Let us represent the set R as the rectangle $ABB'A'$ as in Fig. 2. No point in $\mathrm{Int}(R)$, the interior of R, is equivalent to another point in R. On the other hand, there are pairs of equivalent points on ∂R, the boundary of R. Two points on the line segment AA' are equivalent if they are symmetric with respect to the point O. Similarly, two points on the line segment BB' are equivalent if they are symmetric with respect to the point C. Furthermore, a point on the line segment AB is equivalent to a point on the line segment $A'B'$, if they can be moved to each other by means of the parallel translation given by a vector of length 2 in the direction of the x-axis.

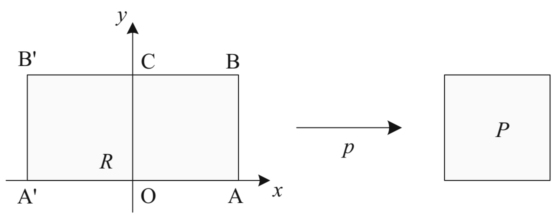

Fig. 2

□

From these observations we conclude that the quotient space P for the equivalence relation given in Definition 1 can be identified with the figure obtained by folding the rectangle R along the line segment OC, and gluing together segments AO and $A'O$, segments BC and $B'C$, and segments AB and $A'B'$. What results is a doubly-covered square.

The following two corollaries can be proved easily.

Corollary 1. *The set of all the lattice points in the plane E can be partitioned into four equivalence classes* {(odd, odd), (odd, even), (even, odd), (even, even)} *depending on the parities of the coordinates, and each of these equivalence classes corresponds by p to each of the four vertices of P, respectively.*

Corollary 2. *Let A and B be an arbitrary pair of lattice points in the plane E, and denote by S the set of all the lattice points lying on the line segment AB. Then $p(S)$ consists of exactly two of the vertices of P. Namely, only two among the four equivalence classes described in Corollary 1 can lie on a straight line.*

2 Developments

Let us consider next what we mean by a development of a doubly-covered square. We will give a definition of a development by using the quotient map $p : E \to P$ introduced in the preceding section. It should be obvious that this definition coincides with the usual definition of a development.

Definition 2. *When a polygon V in the plane E satisfies the following conditions (1) - (3), V is called a* development *of the doubly-covered square P (Fig. 3).*

(1) The map $p|_V : V \to P$, which is the restriction of p to V, is surjective.
(2) $p(\mathrm{Int}(V)) \cap p(\partial V) = \emptyset$.
(3) $p|_{\mathrm{Int}(V)} : \mathrm{Int}(V) \to P$, which is the restriction of p to $\mathrm{Int}(V)$, is injective.

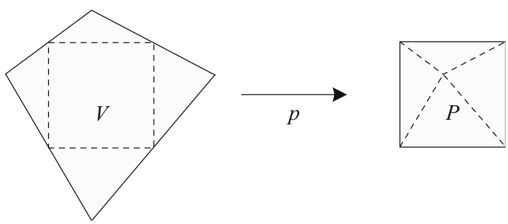

Fig. 3

The lines of folding involved in constructing P from a development V are the lines drawn in the plane E through lattice points parallel to the x- or y-axis.

Associated with this definition, we define as follows the *cut tree* which appears when a doubly-covered square P is developed into a development V.

Definition 3. *Let V be a development of a doubly-covered square P. We call the set $T = p(\partial V)$ the* cut tree *which appears when P is developed into V (Fig. 4).*

According to the results in [4] a cut tree has the following properties:

Lemma 2. *A cut tree T has the following properties:*

(1) T is a tree.
(2) T goes through every vertex of P.
(3) Leaves of T are the vertices of P.

From this lemma, we can get the following corollary easily.

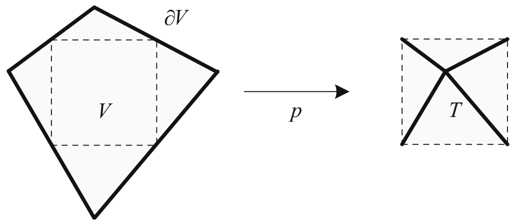

Fig. 4

Corollary 3. *Let V be a development of a doubly-covered square P. Then,*

(1) $\mathrm{Int}(V)$ *contains no lattice points in E.*
(2) For each vertex x of P, there exists at least one lattice point v in V such that $p(v) = x$.

The following lemma follows from Lemma 3.1(4) of paper [3].

Lemma 3. *In order for a point x to be a point of degree d of the cut tree T it is necessary and sufficient that the inverse image $(p|_{\partial V})^{-1}(x)$ of the point x under the map $p|_{\partial V} : \partial V \to P$ consists exactly of d points.*

As a special case of this lemma, we get the following corollary for leaves of a cut tree (points of degree 1):

Corollary 4. *Let a polygon V be a development of a doubly-covered square P with T as its cut tree. Let v be a lattice point lying in ∂V and let $x = p(v)$. Then, the following statements (1) - (3) are mutually equivalent:*

(1) x is a leaf of the cut tree T .
(2) The angle around v within the development V is $180°$.
(3) There exists no other point in ∂V which belongs to the same equivalence class as v.

Proof.
(1) ⇒ (2): Let x be a leaf of T. Since the total angle around the vertex x within P is $180°$, it is clear that there is $180°$ angle around v within the development V.
(2) ⇒ (3): If the total angle around the lattice point v within V is $180°$, then it is impossible to have a development which puts together more angles around v. If there exists another point v' in ∂V which is equivalent to v, then the angle $\theta > 0$ around the vertex v' must be added to the angle around the vertex x of P to make the total angle more than $180°$, which yields a contradiction. Therefore, there cannot be another point in ∂V which is equivalent to v.
(3) ⇒ (1): If there exists no point in ∂V which is equivalent to v, then the inverse image $(p|_{\partial V})^{-1}(x)$ stated in Lemma 3(2) consists exactly of one point. Hence x must be a point of the cup tree T of degree 1, namely, it is a leaf of T. □

3 Faces of Parallelograms

The statement "doubly-covered square P has the top face and the bottom face" sounds plausible, but is it really true? To begin with, is it obvious that a doubly-covered square P consists of two faces? What are these two faces? Let us deal with these questions in this section.

Problem 1. Suppose we have a development $S = S_1 \cup S_2$ consisting of two unit squares sharing a common side as in Fig. 5, and suppose all of the vertices of the two squares S_1 and S_2 correspond under the quotient map p to the vertices of the doubly-covered square P, then we can decompose the doubly-covered square P into two congruent squares $p(S_1)$ and $p(S_2)$. Is such a decomposition unique?

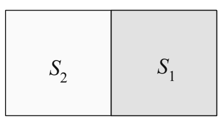

Fig. 5

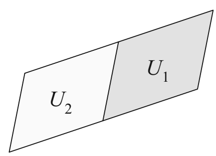

Fig. 6

In this problem, we assumed that all of the vertices of the two squares S_1 and S_2 correspond to the vertices of the doubly-covered square P under the map p. The reason for making this assumption is the fact that only four vertices of the doubly-covered square P have the property that the angle around each of those within the doubly-covered square P is only 180°. Every other point of P has the angle of 360° surrounding it in P.

The answer to Problem 1 above is "Yes, the decomposition is unique". We will call the decomposition $P = p(S_1) \cup p(S_2)$ given uniquely by the two squares the "congruent decomposition by squares". What about the following Problem 2? This is a question posed by replacing squares of Problem 1 by parallelograms.

Problem 2. Suppose we have as in Fig. 6, a development $U = U_1 \cup U_2$ consisting of two congruent parallelograms sharing a common side, and suppose all the vertices of the parallelograms U_1 and U_2 are mapped by the quotient map p onto the vertices of the doubly covered square P, then P can be decomposed into two faces $p(U_1)$ and $p(U_2)$, which are congruent parallelograms. How many such decompositions are there?

We call a decomposition $P = p(U_1) \cup p(U_2)$ into a pair of such congruent parallelograms a *congruent decomposition by parallelograms*. In answer to Problem 2 there are countable infinity of such decompositions. Fig. 7 and 8 illustrate examples of such decompositions. In these figures the diagram on the left gives a development U with two parallelograms U_1 and U_2 distinguished by different colors. The diagram on the right shows the result of constructing the doubly-covered square P by folding the development colored by the two different colors and then developing the result into a rectangle.

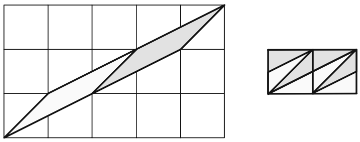

Fig. 7

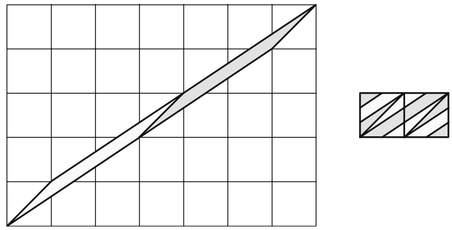

Fig. 8

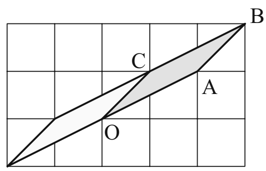

Fig. 9

How can we construct such congruent decompositions by parallelograms? Let us explain the situation by using Fig. 9. Let us first consider a parallelogram $OABC$ of area 1 where vertices are lattice points. (We orient the parallelogram so that $OABC$ refers to the labeling of the vertices in counter clock-wise direction, where O is the origin of the plane E). We will check later the fact that this parallelogram gives a development of a doubly-covered square.

Since the area of the parallelogram is 1, we know by Pick's Theorem that there are no other lattice points beside the vertices in the interior or on the sides of the parallelogram. Therefore, we can take an arbitrary pair of relatively prime integers (a, b) and let $\overrightarrow{OA} = (a, b)$. If the side OA is parallel to the x- or y-axis, then $(a, b) = (\pm 1, 0)$ or $(0, \pm 1)$, respectively.

For $\overrightarrow{OC} = (c, d)$, it suffices to find a pair (c, d) of integers satisfying $ad - bc = 1$ since the area of the parallelogram $OABC$ is equal to 1. Since a and b are relatively prime, we can find a pair c_0, d_0 satisfying $ad_0 - bc_0 = 1$ by using the extended Euclidean Algorithm. Using this pair (c_0, d_0), we let $(c, d) = (c_0 + ak, d_0 + bk)$ for an arbitrary integer k.

The procedure outlined above gives an explicit method for constructing a parallelogram of area 1 with all of its vertices on lattice points. In short, we can say that such a parallelogram can be determined for any choice of integers a, b, c, d satisfying the identity $ad - bc = 1$.

Let us denote by $SL(2, \boldsymbol{Z})$ the set of all 2×2 matrices with integer coefficients having determinant 1. We have the following lemma.

Lemma 4. *Denote by S_1 the unit square with vertices $O(0, 0)$, $A_1(1, 0)$, $B_1(1, 1)$, $C_1(0, 1)$. The following statements (1) and (2) concerning a parallelogram $OABC$ having the origin O as one of the vertices are mutually equivalent.*

(1) The parallelogram $OABC$ has all its vertices on lattice points and has area 1.
(2) The parallelogram $OABC$ is an image of S_1 under some linear transformation given by a matrix belonging to $SL(2, \boldsymbol{Z})$.

Proof. Suppose that the parallelogram $OABC$ has all of its vertices on lattice points and has area 1, and let $\overrightarrow{OA} = (a, b)$ and $\overrightarrow{OC} = (c, d)$. Then, a, b, c, d are all integers and $ad - bc = 1$. Therefore, if we let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the image of the unit square S_1 under the linear transformation f_M given by M is the parallelogram $OABC$. The converse assertion is obvious.

Next, we consider the situation indicated in Fig. 10. The diagram on the left indicates congruent unit squares S_1 and S_2, lying adjacently, while the diagram on the right shows two congruent parallelograms U_1 and U_2 of area 1 lying adjacently and sharing the side OC. By Lemma 4, it is clear that S_1 is mapped onto U_1 by means of a linear transformation f_M given by a matrix M belonging to $SL(2, \boldsymbol{Z})$. It is also obvious that S_2 is mapped onto U_2 by the same linear transformation f_M.

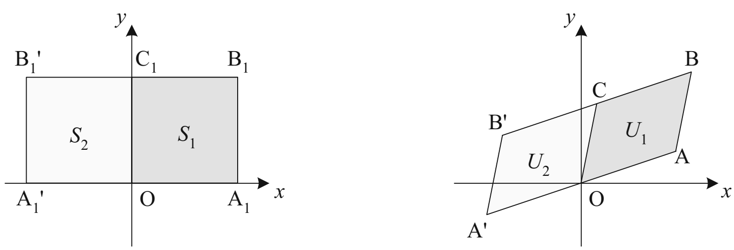

Fig. 10 □

In the next section we will show that the diagram on the right in Fig. 10 gives a development of the doubly-covered square P.

4 Actions of $SL(2, Z)$

The group $SL(2, \boldsymbol{Z})$ acts on the plane $E = \{(x, y) \mid x, y \in \boldsymbol{R}\}$ as a group of linear transformations: $SL(2, \boldsymbol{Z}) \times E \to E$. The next lemma shows that this action preserves the equivalence relation on E given in Definition 1.

Lemma 5. *Let us denote by X_1, X_2 an arbitrary pair of points in E, and denote by $\sim$ the equivalence relation given in Definition 1. Then, for any $M \in SL(2, \boldsymbol{Z})$, $f_M(X_1) \sim f_M(X_2)$ is satisfied if $X_1 \sim X_2$.*

Proof. Suppose $X_1 \sim X_2$, then $X_1 X_2$ satisfy either the condition (1) or condition (2) of Definition 1. We will show that $f_M(X_1) \sim f_M(X_2)$ is satisfied in either case.

Case (1) Suppose that the midpoint of the line segment X_1X_2 is a lattice point L, then, since the transformation f_M is linear and since the components of M are integers, it is clear that the midpoint of the line segment connecting $f_M(X_1)$ and $f_M(X_2)$ is a lattice point, and hence $f_M(X_1) \sim f_M(X_2)$.

Case (2) Suppose that the components of the vector $\overrightarrow{X_1X_2}$ are both even integers, then by the same reason as for case (1), the components of the vector $\overrightarrow{f_M(X_1)f_M(X_2)} = f_M(\overrightarrow{X_1X_2})$ are both even integers, too. Therefore, $f_M(X_1) \sim f_M(X_2)$. □

We obtain from this lemma the following corollary immediately. Note that the map $f_{M^{-1}}$ given by the inverse matrix M^{-1} of M is the inverse map of f_M.

Corollary 5. *For an arbitrary linear transformation f_M given by $M \in SL(2, \boldsymbol{Z})$, there exists a bijective map $\overline{f_M} : P \to P$, which makes the diagram shown in Fig. 11 commutative.*

$$\begin{tikzcd} E \arrow[r, "f_M"] \arrow[d, "p"'] & E \arrow[d, "p"] \\ P \arrow[r, "\overline{f_M}"] & P \end{tikzcd}$$

Fig. 11

From this corollary it follows that the group $SL(2, \boldsymbol{Z})$ acts also on the doubly-covered square $P : SL(2, \boldsymbol{Z}) \times P \to P$. If we incorporate this fact into the discussion of Problem 2 given in the preceding section (cf. Fig. 10), then we see that the diagram shown in Fig. 12 becomes commutative also. Here, $S = S_1 \cup S_2$ and $U = U_1 \cup U_2$.

$$\begin{tikzcd} S \arrow[r, "f_M|_S"] \arrow[d, "p|_S"'] & U \arrow[d, "p|_S"] \\ P \arrow[r, "\overline{f_M}"'] & P \end{tikzcd}$$

Fig. 12

Theorem 1. *Let O be the origin of E, and denote by U_1 a parallelogram $OABC$ of area 1 with all of its vertices lying on lattice points. Let U_2 be a parallelogram $OCB'A'$ congruent to U_1 and lying adjacent to U_1 sharing the side OC. Then $U = U_1 \cup U_2$ is a development of P.*

Proof. We show that the properties (1) - (3) of Definition 2 are satisfied.

(1) $p|_U$ is surjective, since both $p|_S$ and $\overline{f_M}$ are surjective.
(2) Both of the two horizontal maps $f_M|_S$ and $\overline{f_M}$ in the commutative diagram above are bijective, and hence we have $p(\mathrm{Int}(U)) \cap p(\partial U) = \emptyset$ since $p(\mathrm{Int}(S)) \cap p(\partial S) = \emptyset$.
(3) We obtain the commutative diagram shown in Fig. 13 by restricting the commutative diagram above to $\mathrm{Int}(S)$, the interior of S, and $\mathrm{Int}(U)$, the interior of U.

$$\begin{tikzcd}
\mathrm{Int}(S) \arrow[r, "f_M|_{\mathrm{Int}(S)}"] \arrow[d, "p|_{\mathrm{Int}(S)}"'] & \mathrm{Int}(U) \arrow[d, "p|_{\mathrm{Int}(U)}"] \\
P \arrow[r, "\overline{f_M}"'] & P
\end{tikzcd}$$

Fig. 13

Since both of the maps $f_M|_{\mathrm{Int}(S)}$ and $\overline{f_M}$ are bijections, and since $p|_{\mathrm{Int}(S)}$ is injective, we see that $p|_{\mathrm{Int}(U)}$ is also injective. □

From the discussions above, we conclude that the set of all decompositions of the doubly-covered square P into two congruent parallelograms, which answers Problem 2 is given by $\{\, (p(f_M(S_1)),\, p(f_M(S_2))) \mid M \in SL(2, \boldsymbol{Z})\}$.

5 Convex Developments

We shall consider convex developments of the doubly-covered square P. In [2] we determined all the convex developments of P obtained by developing along a cut tree T, which appears only on the square face S_2 (Fig. 14). We did not discuss in [2] the case of developments with cut trees appearing on both of the faces S_1 and S_2 such as those indicated in Fig. 15.

The purpose of our discussion in this section is to extend our results in [2] to the case of general convex developments (where cut trees T may appear on both of the faces S_1 and S_2 of P). Let us denote by V an arbitrary convex development of P (Fig. 16). The interior of V contains no lattice points because of Corollary 3(1). According to Corollary 3(2) there are several ways of choosing four lattice points A_0, A_1, A_2, A_3 on the boundary ∂V, which correspond under the quotient map p to the four vertices of P. Let us choose one such quadruple A_0, A_1, A_2, A_3, and let us denote by W the convex closure in E determined by these four points.

Lemma 6. *For the convex closure W determined by the four lattice points A_0, A_1, A_2, A_3, the following assertions are valid.*

(1) W is a convex quadrilateral.
(2) There are no other lattice points in the interior or on the sides of W.

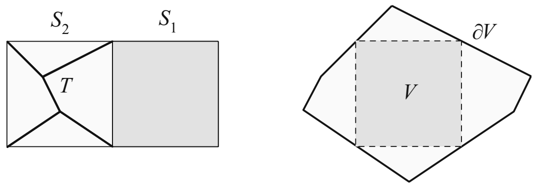

Fig. 14

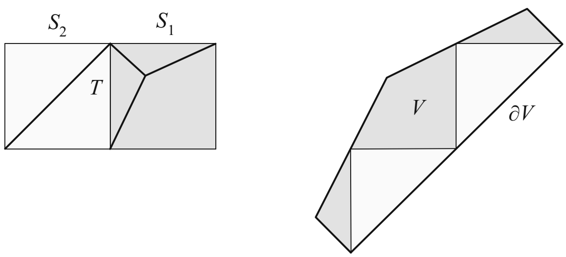

Fig. 15

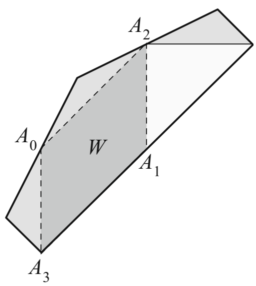

Fig. 16

(3) The area of W equals 1.
(4) W is a parallelogram.

We will postpone the proof of Lemma 6 till the end of this section. In the sequel we shall assume that one of the vertices of the convex closure W is the origin O of E, by translating W in the x- and/or y-axis direction by the amount of even integers, if necessary. By Lemma 6 above, we can assume that the convex closure W is a parallelogram of area 1 with all of its vertices lying on lattice points. Let $U_1 = W$. Then, by Lemma 4, the parallelogram U_1 is given as $U_1 = f_M(S_1)$ by means of some matrix $M \in SL(2, \boldsymbol{Z})$. By defining a convex polygon V_S by $V_S = (f_M)^{-1}(V)$, we obtain the commutative diagram shown in Fig. 17.

Lemma 7. *V_S is a development of P.*

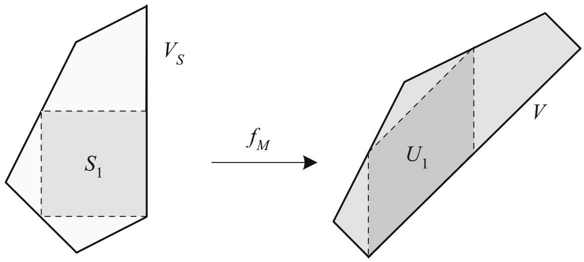

Fig. 17

Proof. We show that the properties (1) - (3) in Definition 2 are satisfied. Since both of the horizontal maps $f_M|_{V_S}$, $\overline{f_M}$ in the commutative diagram above are bijections and since $p|_V$ is surjective, we see that the map $p|_{V_S}$ is surjective. Similarly, from the fact that $p(\mathrm{Int}(V)) \cap p(\partial V) = \emptyset$, it follows that $p(\mathrm{Int}(V_S)) \cap p(\partial V_S) = \emptyset$. The fact that $p|_{\mathrm{Int}(V_S)} : \mathrm{Int}(V_S) \to P$ is injective follows similarly from the fact that $p|_{\mathrm{Int}(V)} : \mathrm{Int}(V) \to P$ is injective. Thus, V_S is a development of P. □

Let the unique congruent decomposition of the doubly-covered square P by squares be denoted by $P = p(S_1) \cup p(S_2)$. Since $\mathrm{Int}(U_1) \cap \partial V = \emptyset$ in Fig. 17, we see that $\mathrm{Int}(S_1) \cap \partial V_S = \emptyset$ is also satisfied. Therefore, the cut tree $T_S = p(\partial V_S)$ appearing when P is developed into V_S must satisfy $p(\mathrm{Int}(S_1)) \cap T_S = \emptyset$. From this fact, we see that the inclusion $T_S \subset p(S_2)$ must take place, and hence the cut tree T_S of the development V_S must be contained only in one face S_2 (Fig. 18). We investigated fully in [2] the properties of such a development V_S. Therefore, we can conclude that the following Theorem 2 is valid. Before we state the theorem, let us introduce some notations. Denote by $\mathcal{V}$ the set of all convex developments of the doubly-covered square P. We also denote by $\mathcal{V}_S$ the subset of $\mathcal{V}$ consisting of those convex developments whose cut tree appears only in one face S_2 in the congruent decomposition by two squares of P.

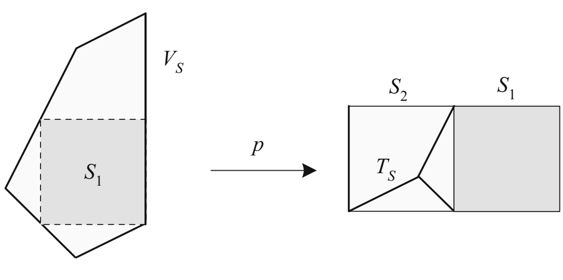

Fig. 18

Theorem 2. *For an arbitrary convex development $V \in \mathcal{V}$, there exist a convex development $V_S \in \mathcal{V}_S$ and a matrix $M \in SL(2, \boldsymbol{Z})$ such that V and $f_M(V_S)$ are congruent.*

Finally, let us give the proof of Lemma 6.

Proof of Lemma 6

(1) By Corollary 1, the set of all lattice points in E is decomposed into four equivalence classes. The four lattice points A_0, A_1, A_2, A_3 are chosen from distinct equivalence classes. By Corollary 2, no three points among these four lattice points lie on a straight line. Therefore, the convex closure W must either be a triangle or a convex quadrilateral. If W were a triangle, then one of the four lattice points must lie in the interior of W. This would imply that the convex development V must have a lattice point in its interior, which contradicts Corollary 3(1). Thus, W must be a convex quadrilateral.

(2) If there is another lattice point in the interior of W then the interior of the development V must have a lattice point, which is a contradiction. So, suppose there is a lattice point B lying on the side A_iA_j of W, distinct from A_i or A_j. In this case, the side A_iA_j contains three lattice points. By Corollary 2, B must belong to the same equivalence class with either A_i or A_j. Furthermore, the point B must lie on the boundary ∂V of the convex polygon V, and that would imply that the line segment A_iA_j must be contained in ∂V. This shows that the angle around B in V must be 180°. But, Corollary 4 then implies that there are no other lattice points in V belonging to the same equivalence class, which yields a contradiction.

(3) If we apply Pick's Theorem to the fact shown in (2) above, we can conclude that the area of W is 1.

(4) Let A_0A_i be a diagonal of W, and let A_j, A_k be the other lattice points of W. Let A'_k be the point which makes $A_0A_jA_iA'_k$ a parallelogram. Let us show that $A_k = A'_k$. Let l be the line parallel to the diagonal A_0A_i and going through the point A_k (Fig. 19). Since the areas of the two triangles $\triangle A_0A_iA_k$, $\triangle A_0A_iA'_k$ are both $\frac{1}{2}$, we see that the point A'_k must lie on the line l. The set of all the lattice points lying on the line l can be represented as $\{X : A_0X = A_0A'_k + nA_0A_i,\ n \in \boldsymbol{Z}\}$. There is only one point X from this set for which $A_0A_jA_iX$ is a convex quadrilateral; it is when $X = A'_k$, and then, $A_0A_jA_iA'_k$ is a parallelogram. Therefore, we must have $A_k = A'_k$.

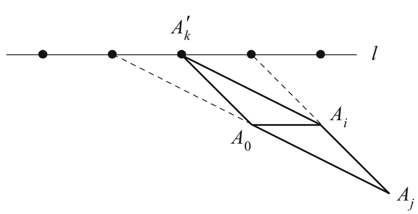

Fig. 19

□

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Flat 2-Foldings of Convex Polygons

Jin Akiyama[1], Koichi Hirata[2], Mari-Jo P. Ruiz[3], and Jorge Urrutia[4,⋆]

[1] Research Institute of Educational Development Tokai University,
Shibuya-ku, Tokyo, Japan 151-8677
fwjb5117@mb.infoweb.ne.jp

[2] Department of Mathematics, Faculty of Education, Ehime University,
3 Bunkyo-cho, Matsuyama, Japan 790-8577
hirata@ed.ehime-u.ac.jp

[3] Mathematics Department, School of Science and Engineering,
Ateneo de Manila University, Quezon City, Philippines
mruiz@ateneo.edu

[4] Instituto de Matemáticas, Universidad Nacionál Autónoma de México,
México D.F. México
urrutia@math.unam.mx

Abstract. A folding of a simple polygon into a convex polyhedron is accomplished by glueing portions of the perimeter of the polygon together to form a polyhedron. A polygon Q is a *flat n-folding* of a polygon P if P can be folded to exactly cover the surface of Q n times, with no part of the surface of P left over. In this paper we focus on a specific type of flat 2-foldings, flat 2-foldings that *wrap* Q; that is, foldings of P that cover both sides of Q exactly once. We determine, for any n, all the possible flat 2-foldings of a regular n-gon. We finish our paper studying the set of polygons that are flat 2-foldable to regular polygons.

1 Introduction

A folding of a simple polygon into a convex polyhedron is accomplished by glueing portions of the perimeter of the polygon together to form the polyhedron (Fig. 1). The paper [1] proves the existence of nondenumerably infinite foldings of simple polygons to convex polyhedra. In [2, 5, 3, 4], all possible foldings of an equilateral triangle, square, regular pentagon, and regular n-gons, respectively, are determined, $n \geq 6$. This paper deals with related constructions. Let P and Q be two polygons. We say that Q is a *flat n-folding* of a polygon P if P can be folded to exactly cover the surface of Q n times, with no part of the surface of P left over. In this paper we focus on a specific type of flat 2-foldings, flat 2-foldings that *wrap* Q; that is, foldings of P that cover both sides of Q exactly once. We use *flat 2-folding* as the meaning of such a flat 2-folding in this paper. For example, in Fig. 1 we show a flat 2-folding of a regular hexagon (that wraps an equilateral triangle).

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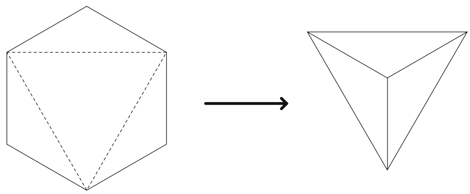

Fig. 1. Folding a regular hexagon to wrap an equilateral triangle

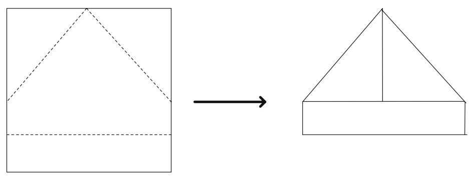

Fig. 2. A flat 2-folding of a square to a pentagon

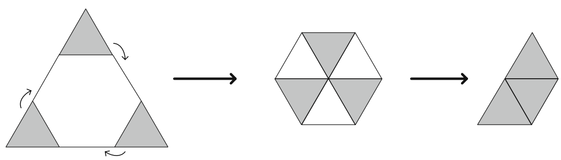

Fig. 3. A flat 3-folding of an equilateral triangle to a trapezoid

In Section 2, we determine all the convex polygons that can result from flat 2-foldings of regular polygons. In Section 3, we determine all the convex polygons that can be flat 2-folded to regular polygons. We conclude the paper with some remarks and open problems.

2 Flat 2-Foldings of Regular n-Gons

2.1 The Regular Pentagon and the Regular n-Gons, $n \geq 7$

Let P_n be a regular n-gon. The interior angle at a vertex of P_n is $\Theta_n = \pi - \frac{2\pi}{n}$. In a flat 2-folding of P_n, a necessary condition for a vertex v of P_n to coincide with an interior point q of Q is that there exist some positive integer $m \leq n$, such that $m\Theta_n = 2\pi$ or $m\Theta_n = \pi$ (Fig. 4). If $m\Theta_n = \pi$, then P_n has an edge that is incident with vertex v in the flat 2-folding. These inequalities cannot be satisfied for $n = 5$, or any $n \geq 7$. This proves the following:

Proposition 1. *Any flat 2-folding of a regular $(2n + 1)$-gon, $n \geq 2$, can be obtained by folding along one of its lines of symmetry. Flat 2-foldings of a regular $2n$-gons, $n \geq 4$, can be obtained by folding either along a line of symmetry that bisects two opposite sides or one that bisects two opposite angles.*

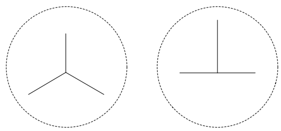

Fig. 4. Here $3\Theta_3 = 2\pi$, and $2\Theta_4 = \pi$

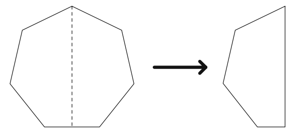

Fig. 5. A flat 2-folding of P_7

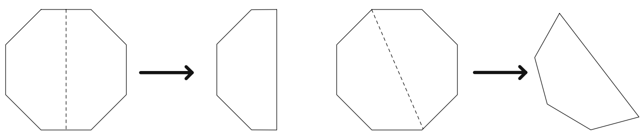

Fig. 6. Flat 2-foldings of P_8

Fig. 5 and 6 provide an illustration of Proposition 1.

2.2 The Regular Hexagon

One way to obtain a flat 2-folding of a regular hexagon is to fold along one of its lines of symmetry. If any other flat 2-foldings exist, then the necessary condition mentioned earlier must be satisfied. Certainly $3\Theta_6 = 2\pi$, and in fact two other ways to obtain a flat 2-folding can be found:

1. Choose three alternate vertices of the hexagon. The other three vertices determine an equilateral triangle. Fold the hexagon along the sides of this triangle so that the three chosen vertices meet at the center of the hexagon.
2. Choose two adjacent sides of the hexagon and their opposite sides. The midpoints of these four sides determine a rectangle. Fold the hexagon along the sides of this rectangle so that each set of three consecutive vertices enclosing the chosen adjacent sides meets at a point.

It is easy to check that no other flat 2-foldings exist. This proves the following.

Proposition 2. *The foldings shown in Fig. 7 are all the possible flat* 2*-foldings of a regular hexagon.*

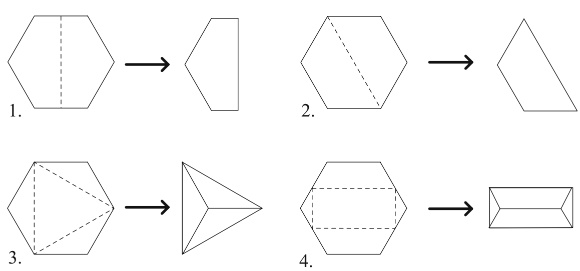

Fig. 7. All possible flat 2-foldings of a regular hexagon

2.3 The Square

To obtain flat 2-foldings of a square, we can again fold along lines of symmetry. Alternatively, we may note that the necessary condition is satisfied: $4\Theta_4 = 2\pi$ and $2\Theta_4 = \pi$, and search for ways in which the four vertices of the square can coincide with an interior point of the square or two vertices of the square can coincide with an interior point and be incident with a side of the square. If any other flat 2-foldings exist, each vertex of the square must coincide with a point on a side of the square. Such foldings are obtained as follows. Choose two parallel lines in the interior of the square such that the lines are a distance $\frac{1}{2}l$ apart, where l is the length of the side of the square and fold the square along these lines. These considerations lead to the following proposition.

Proposition 3. *The foldings shown in Fig. 8 are all the possible flat 2-foldings of a square.*

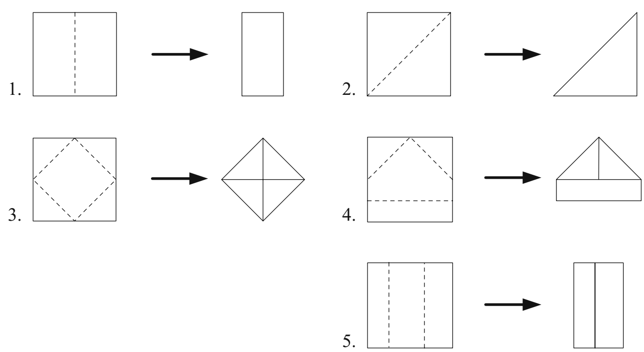

Fig. 8. All possible flat 2-foldings of a square

2.4 The Equilateral Triangle

A flat 2-folding of an equilateral triangle can be obtained by folding along a line of symmetry. Although the necessary condition is satisfied: $3\Theta_3 = \pi$, there is no way that the three vertices of the triangle can meet at an interior point and be incident with a side of the triangle. Hence, if any other flat 2-foldings of the triangle exist, each vertex of the triangle must coincide with a point on a side of the triangle. In fact, this point must be the midpoint of a side; otherwise, a flat 2-folding will not be possible.

If the point is the midpoint of the side opposite the vertex, then the remaining uncovered surface areas will consist of equilateral triangles (Fig. 9). These equilateral triangles can be folded into themselves in three essentially different ways (Fig. 10).

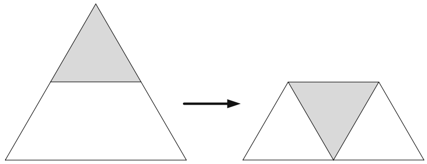

Fig. 9

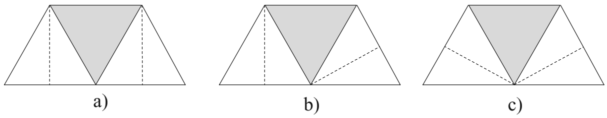

Fig. 10

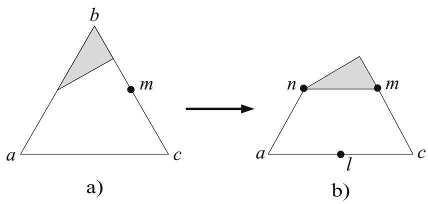

Fig. 11

If the point is the midpoint of an adjacent side, the result is the configuration shown in Fig. 11b). Consider the vertex c. If c is made to coincide with the midpoint m, then the resultant folding is the same as that of Fig. 10a). If c is made to coincide with l, then the folding that results is essentially the same as that of Fig. 10b). If c is made to coincide with n, then the folding that results will either be the same as that of Fig. 10b) or c), according to whether a is made to coincide with n or l.

The possibilities for vertex a can be identified in the same way. They will be included among the foldings shown in Fig. 10.

Hence only four different flat 2-foldings can be obtained from an equilateral triangle. This proves the following proposition.

Proposition 4. *The foldings shown in Fig. 12 are all the possible flat* 2*-foldings of an equilateral triangle.*

3 Convex Polygons Flat 2-Foldable to Regular Polygons

In the previous section, we answered the question of what convex polygons can result from flat 2-foldings of regular polygons? In this section, we turn the question around; what convex polygons can be flat 2-folded to regular polygons ?

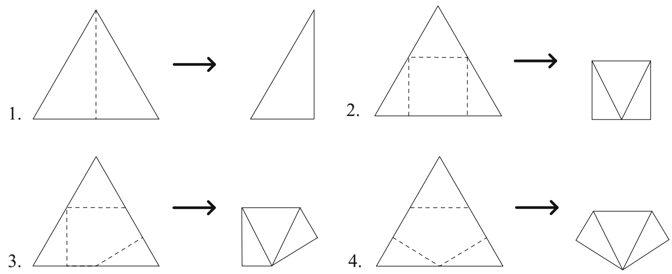

Fig. 12

3.1 Regular Polygons with at Least Five Vertices

We first prove that no convex polygon is flat 2-foldable to a regular n-gon, $n \geq 5$. Let P be a convex polygon with vertices $\{p_1, \ldots, p_n\}$. For each i, let α_i be the internal angle of P at vertex p_i and let $t(p_i) = \pi - \alpha_i$ (Fig. 13).

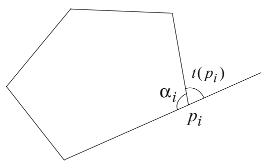

Fig. 13

Observation 1: $\sum_{i=1}^{n} t(p_i) = 2\pi$.

Let Q be a flat 2-folding of P, and $\{q_1, \ldots, q_m\}$ and $\{\beta_1, \ldots, \beta_m\}$ the vertices and angles of Q. Consider an angle $\beta_i > \frac{\pi}{2}$ of Q. Since Q is a flat 2-folding of P, one or more vertices of P are mapped to q_i.

Case 1: Exactly one vertex p_j of P is mapped onto q_i. This case happens when q_i was obtained by folding P along an edge of Q and p_j is mapped to q_i (Fig. 14). In this case $t(p_j) = 2t(q_i)$.

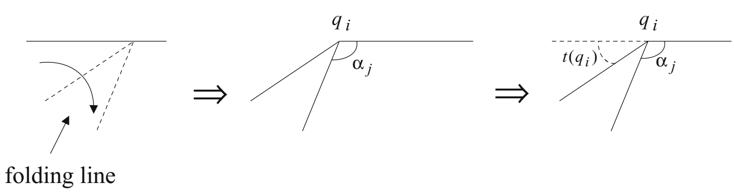

Fig. 14

Case 2: Suppose that $k \geq 2$ vertices $p_1, \ldots, p_k$ of P are mapped to q_i. Observe that

$$\sum_{j=1}^{k} \alpha_j \leq 2\beta_i.$$

Since

$$\sum_{j=1}^{k} t(p_j) = \sum_{j=1}^{k} (\pi - \alpha_j) = k\pi - \sum_{j=1}^{k} \alpha_j \geq k\pi - 2\beta_i \geq 2(\pi - \beta_i) = 2t(q_i),$$

we have proved the following.

Lemma 1. *Let Q be a flat 2-folding of P and q_i a vertex of Q such that $\beta_i > \frac{\pi}{2}$. Then if $p_1, \ldots, p_k$ are mapped to q_i,*

$$\sum_{j=1}^{k} t(p_j) \geq 2t(q_i).$$

Theorem 1. *No convex polygon is flat 2-foldable to a regular n-gon, $n \geq 5$.*

Proof. Let Q_n be a regular n-gon, $n \geq 5$. Then all internal angles of Q_n are greater than $\frac{\pi}{2}$. Observe that

$$\sum_{i=1}^{n} t(q_i) = 2\pi.$$

Suppose that Q_n is a flat 2-folding of some convex polygon P. For each i, let S_i be the set of vertices of P mapped in the folding to q_i, $1 \leq i \leq n$. Observe that $S_i \cap S_j = \emptyset$, $i \neq j$. By Lemma 1,

$$t(S_i) = \sum_{p_j \in S_i} t(p_j) \geq 2t(q_i),$$

and thus

$$\sum_{i=1}^{n} t(S_i) \geq 2 \sum_{i=1}^{n} t(q_i) = 4\pi,$$

which is a contradiction. □

In view of the theorem, we now proceed to study the set of convex polygons flat 2-foldable to equilateral triangles and squares.

3.2 The Equilateral Triangle

A way to obtain convex polygons flat 2-foldable to an equilateral triangle is as follows. Take an equilateral triangle T and a point p on it. For each edge e_i of T, let p_i be the mirror image of p with respect to the line generated by e_i (Fig. 15).

Let P be the polygon whose vertices are p_1, p_2, p_3 and the vertices of T. Depending on the position of p, we obtain a hexagon (Fig. 15), a pentagon, or a quadrilateral (Fig. 16). It is now not difficult to see that with this procedure, all polygons flat 2-foldable to an equilateral triangle will be generated.

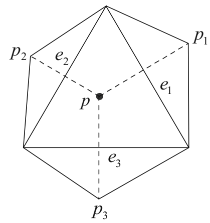

Fig. 15

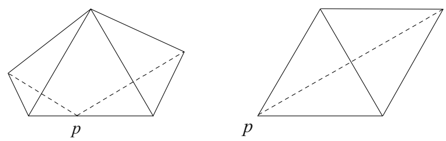

Fig. 16

3.3 The Square

Let us take a square S as Q. There are two types of polygons which are flat 2-foldable to a square; those that contain a copy of S (which is not folded in the flat 2-folding of P) (Fig. 17a)), and those which do not contain such a copy of S (Fig. 17b)). In this paper we confine ourselves to foldings of the first type and characterize them. The second type of flat 2-foldings to squares will be characterized in a forthcoming paper.

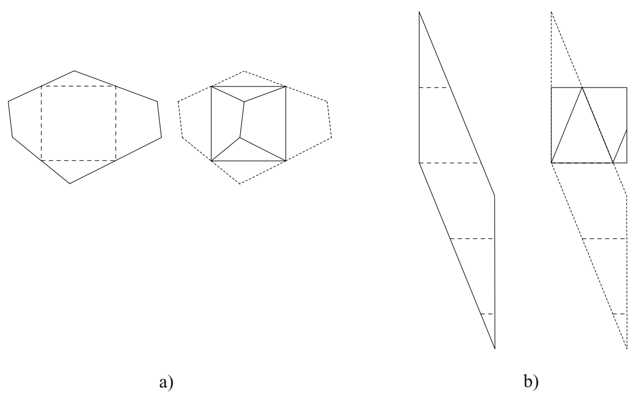

Fig. 17

Let Q be a convex polygon and P a convex polygon flat 2-foldable to Q. Consider a flat 2-folding of P to Q. A point q of Q, *not* a vertex of Q, is called *singular* if in the folding of P to Q, at least one vertex of P is mapped to q.

There are two types of singular points, those lying in the interior of Q, and those lying in the relative interior of edges of Q. The following three lemmas are given without proof.

Lemma 2. *Let S_i be the set of vertices of P mapped to a singular point q of Q. Then*

$$\sum_{p_j \in S_i} t(p_j) \geq \pi.$$

Lemma 3. *In a flat 2-folding of P to Q, Q has at most two singular points.*

Let q be a singular point of Q such that k vertices of P are mapped to Q. We call k the degree of q.

Lemma 4. *The degree of any singular point of Q is at most four. Moreover if Q has a singular point q of degree four, then q is the only singular point of Q.*

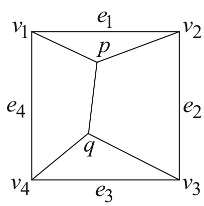

Fig. 18

Consider the labeling on the vertices and edges of S as in Fig. 18, and p, q, two points on S, *not necessarily different*. Suppose that we join p to v_1, v_2 and q, and q to v_3 and v_4 using non-crossing line segments. We also require that the angles formed around p and q by these line segments be less that or equal to π, as shown in Fig. 18. Let p_1, p_2 and p_4 be the mirror images of p with respect to e_1, e_2, and e_4. Define q_2, q_3, and q_4 similarly; see Fig. 19.

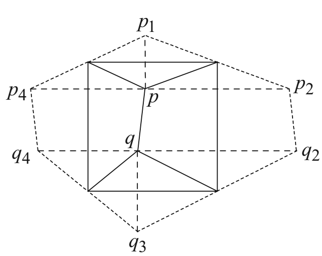

Fig. 19

Clearly the polygon with vertex set $\{p_1, p_2, q_2, q_3, q_4, p_4\}$ is flat 2-foldable to S. According to the position of p and q, we obtain the cases shown in Fig. 20, which characterize the set of all polygons of the first type that are flat 2-foldable to a square. It is now not difficult to show, using the preceding lemmas, that any convex polygon flat 2-foldable to S can be obtained from one of the cases shown in Fig. 20.

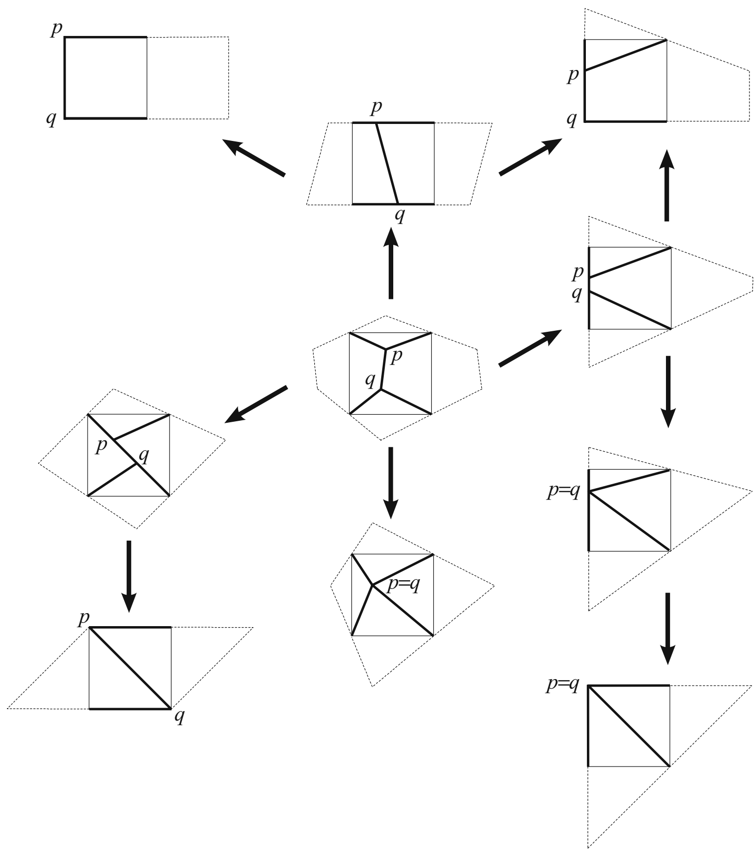

Fig. 20

4 Further Research

The problem of determining all flat n-foldable convex polygons $n \geq 2$ remains open. In a forthcoming paper, we identify all convex polygons that are flat 2-foldable to a square, and those that are flat 3-foldable to a triangle.

A more specific question is the following. Are there any convex polygons other than the rectangle which are flat n-foldable for any n?

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Uniform Coverings of 2-Paths with 6-Paths in the Complete Graph

Jin Akiyama[1], Midori Kobayashi[2,⋆], and Gisaku Nakamura[3]

[1] Research Institute of Educational Development,
Tokai University, Shibuya-ku Tokyo 151-0063, Japan
[2] School of Administration and Informatics,
University of Shizuoka, Shizuoka 422-8526, Japan
[3] Research Institute of Educational Development,
Tokai University, Shibuya-ku Tokyo 151-0063, Japan

Abstract. Let $n \geq 7$. Then there exists a uniform covering of 2-paths with 6-paths in K_n if and only if $n \equiv 0, 1, 2 \pmod 5$.

1 Introduction

Let K_n be the complete graph on n vertices. A k-path is a path of length k and a k-cycle is a cycle of length k, where the length of a path [cycle] is the number of edges in the path [cycle].

A uniform covering of the 2-paths in K_n with k-paths [k-cycles] is a set S of k-paths [k-cycles] having the property that each 2-path in K_n lies in exactly one k-path [k-cycle] in S. Only the following cases of the problem of constructing a uniform covering of the 2-paths in K_n with k-paths or k-cycles have been solved:

1. with 3-cycles,
2. with 3-paths [2],
3. with 4-cycles [3],
4. with 4-paths [5],
5. with 5-paths [6,7],
6. with 6-cycles [8],
7. with n-cycles (Hamilton cycles) when n is even [4].

In this paper, we solve the problem in the case of 6-paths, that is, we prove:

Theorem 1.1 *Let $n \geq 7$. Then there exists a uniform covering of 2-paths with 6-paths in K_n if and only if $n \equiv 0, 1, 2 \pmod 5$.*

2 The Case n is Small

In this section, we construct a uniform covering of 2-paths with 6-paths in K_n when n is small.

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Proposition 2.1 *There exists uniform coverings of* 2*-paths with* 6*-paths in* K_n *when* $n = 7, 10, 11, 12, 15, 16, 17$.

Proof. Let $V_n = \{0, 1, 2, \ldots, n-1\}$ be the vertex set of K_n. We call a uniform covering of 2-paths with Hamilton cycles in K_n a Dudeny set in K_n. It is known that there exists a Dudeny set in K_n when $3 \leq n \leq 20$ [1]. We denote it by $\mathcal{D}_n$.

(1) $n = 7$

For a Hamilton cycle in $\mathcal{D}_8$, $H = (7, a_1, a_2, \ldots, a_7)$, define a 6-path H^*: $H^* = [a_1, a_2, \ldots, a_7]$. Then $\{H^* | H \in \mathcal{D}_8\}$ is a uniform covering of 2-paths with 6-paths in K_7.

(2) $n = 10$

For a Hamilton cycle in $\mathcal{D}_{10}$, $H = (0, a_1, a_2, \ldots, a_9)$, where $a_1 < a_9$, define two 6-paths H^*, H^{**}: $H^* = [0, a_1, a_2, a_3, a_4, a_5, a_6]$, $H^{**} = [a_5, a_6, a_7, a_8, a_9, 0, a_1]$. Then $\{H^*, H^{**} | H \in \mathcal{D}_{10}\}$ is a uniform covering of 2-paths with 6-paths in K_{10}.

(3) $n = 11$

Define σ_{11}: $\sigma_{11} = (0\ 1\ 2\ \cdots\ 10)$ so that σ_{11} is a vertex-rotation in K_{11}. Put $P_1 = [2, 8, 3, 7, 4, 6, 5]$, $P_2 = [10, 1, 8, 3, 6, 5, 4]$, $P_3 = [4, 7, 1, 10, 9, 2, 6]$, $P_4 = [9, 2, 5, 6, 1, 10, 8]$, $P_5 = [10, 7, 4, 2, 9, 8, 3]$, $P_6 = [0, 10, 1, 9, 2, 7, 4]$, $P_7 = [0, 9, 2, 1, 10, 4, 7]$, $P_8 = [1, 10, 5, 6, 9, 2, 8]$, $P_9 = [8, 3, 2, 9, 7, 4, 5]$. Then $\{\sigma_{11}^i P_j | 0 \leq i \leq 10, 1 \leq j \leq 9\}$ is a uniform covering of 2-paths with 6-paths in K_{11}.

(4) $n = 12$

For a Hamilton cycle in $\mathcal{D}_{13}$, $H = (12, a_1, a_2, \ldots, a_{12})$, where $a_1 < a_{12}$, define two 6-paths H^*, H^{**}: $H^* = [a_1, a_2, a_3, a_4, a_5, a_6, a_7]$, $H^{**} = [a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}]$. Then $\{H^*, H^{**} | H \in \mathcal{D}_{13}\}$ is a uniform covering of 2-paths with 6-paths in K_{12}.

(5) $n = 15$

For a Hamilton cycle in $\mathcal{D}_{15}$, $H = (0, a_1, a_2, \ldots, a_{14})$, where $a_1 < a_{14}$, define three 6-paths H^*, H^{**}, H^{***}: $H^* = [0, a_1, a_2, a_3, a_4, a_5, a_6]$, $H^{**} = [a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}]$, $H^{***} = [a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, 0, a_1]$. Then $\{H^*, H^{**}, H^{***} | H \in \mathcal{D}_{15}\}$ is a uniform covering of 2-paths with 6-paths in K_{15}.

(6) $n = 16$

Define σ_{16}: $\sigma_{16} = (0\ 1\ 2\ \cdots\ 15)$ so that σ_{16} is a vertex-rotation in K_{16}. Put $P_1 = [6, 9, 1, 0, 15, 7, 10]$, $P_2 = [5, 10, 2, 0, 14, 6, 11]$, $P_3 = [0, 4, 12, 2, 10, 14, 15]$, $P_4 = [7, 3, 12, 0, 4, 13, 9]$, $P_5 = [2, 12, 5, 0, 11, 4, 14]$, $P_6 = [4, 1, 10, 0, 6, 15, 12]$, $P_7 = [4, 14, 9, 0, 7, 15, 6]$, $P_8 = [1, 11, 13, 0, 3, 5, 15]$, $P_9 = [11, 0, 9, 12, 5, 7, 14]$, $P_{10} = [2, 0, 9, 10, 1, 15, 4]$, $P_{11} = [3, 2, 8, 7, 1, 0, 9]$, $P_{12} = [5, 6, 11, 10, 15, 0, 9]$, $P_{13} = [10, 0, 11, 1, 6, 2, 13]$, $P_{14} = [3, 4, 0, 6, 10, 7, 11]$, $P_{15} = [5, 11, 13, 3, 6, 0, 12]$, $P_{16} = [1, 12, 9, 4, 0, 5, 2]$, $P_{17} = [5, 1, 2, 6, 0, 3, 9]$, $P_{18} = [3, 0, 5, 7, 12, 10, 11]$, $P_{19} = [8, 6, 10, 12, 0, 14, 13]$, $P_{20} = [5, 2, 4, 3, 0, 1, 14]$, $P_{21} = [5, 7, 4, 0, 13, 12, 10]$. Then $\{\sigma_{16}^i P_j | 0 \leq i \leq 15, 1 \leq j \leq 21\}$ is a uniform covering of 2-paths with 6-paths in K_{16}.

(7) $n = 17$

For a Hamilton cycle in $\mathcal{D}_{18}$, $H = (17, a_1, a_2, \ldots, a_{17})$, where $a_1 < a_{17}$, define three 6-paths H^*, H^{**}, H^{***}: $H^* = [a_1, a_2, a_3, a_4, a_5, a_6, a_7]$, $H^{**} = [a_6, a_7, a_8, a_9,$

$a_{10}, a_{11}, a_{12}]$, $H^{***} = [a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}]$. Then $\{H^*, H^{**}, H^{***} | H \in \mathcal{D}_{18}\}$ is a uniform covering of 2-paths with 6-paths in K_{17}. □

3 Main Proposition

Proposition 3.1 *Let $m \geq 10$. If there exists a uniform covering of 2-paths with 6-paths in K_m, then there exists a uniform covering of 2-paths with 6-paths in K_{m+10}.*

Proof. Let V_m, V_{10} and V be the vertex sets of K_m, K_{10} and K_{m+10}, respectively. Put $V_m = \{0, 1, 2, \ldots, m-1\} = \mathbf{Z}_m$ (the set of integers modulo m), $V_{10} = \{a, b, c, d, e, f, g, h, i, j\}$ and $V = V_m \cup V_{10}$.

We denote by σ the vertex-permutation of K_{m+10}: $\sigma = (a)(b\ c\ d\ e\ f\ g\ h\ i\ j)$. We denote by τ the vertex-permutation of K_{m+10}: $\tau = (0\ 1\ 2\ \cdots\ m{-}1)$. Note that σ acts on V_m trivially, i.e., as the identity map, and τ acts on V_{10} trivially, and we have $\sigma\tau = \tau\sigma$. Put $\Sigma = \{\sigma^s | 0 \leq s \leq 8\}$, and $T = \{\tau^t | 0 \leq t \leq m-1\}$. For a set of paths $\mathcal{C}$ in K_{m+10}, define $\Sigma\mathcal{C} = \{\sigma^s P | P \in \mathcal{C}, 0 \leq s \leq 8\}$ and $T\mathcal{C} = \{\tau^t P | P \in \mathcal{C}, 0 \leq t \leq m-1\}$. For a set of paths $\mathcal{C}$, we denote by $\pi(\mathcal{C})$ the set of the 2-paths which belong to $\mathcal{C}$.

For any edge $\{x, y\}$ in K_m, we define the length $d(x, y)$:

$$d(x, y) = y - x \pmod{m}.$$

For any two lengths d_1, d_2, we define that d_1 and d_2 are equal as lengths when $d_1 = d_2$ or $d_1 = -d_2 \pmod{m}$.

There exist uniform coverings of 2-paths with 6-paths in K_m and K_{10}, so the 2-paths in K_m and the 2-paths in K_{10} are covered with them. The set of 2-paths in K_{m+10} which don't belong to K_m or K_{10} is $\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4$, where,

$$\begin{aligned}
\Pi_1 &= \{[u, x, v] \mid u, v \in V_{10}, u \neq v, x \in V_m\}, \\
\Pi_2 &= \{[u, v, x] \mid u, v \in V_{10}, u \neq v, x \in V_m\}, \\
\Pi_3 &= \{[x, u, y] \mid u \in V_{10}, x, y \in V_m, x \neq y\}, \\
\Pi_4 &= \{[u, x, y] \mid u \in V_{10}, x, y \in V_m, x \neq y\}.
\end{aligned}$$

(I) Construction of a set of 6-paths $\mathcal{C}$ in K_{m+10} such that $\pi(\mathcal{C}) = \Pi_1 \cup \Pi_2$.
Put

$$\begin{aligned}
P &= [0, c, a, 1, b, e, 2], \\
Q_1 &= [j, 0, c, d, 1, i, e], \\
Q_2 &= [h, 0, e, g, 1, f, j],
\end{aligned}$$

and put $\mathcal{P} = T\Sigma\{P\}$, $\mathcal{Q} = T\Sigma\{Q_1, Q_2\}$.

Claim 3.1 $\pi(\mathcal{P} \cup \mathcal{Q}) = \Pi_1 \cup \Pi_2$.

Proof. It is trivial that $\pi(\mathcal{P} \cup \mathcal{Q}) \subset \Pi_1 \cup \Pi_2$. We will show that $\pi(\mathcal{P} \cup \mathcal{Q}) \supset \Pi_1 \cup \Pi_2$.

(1) Consider a 2-path $S = [u, x, v]$, $u, v \in V_{10}, u \neq v, x \in V_m$.

If $u = a$ or $v = a$, we may consider $u = a$ as $[u, x, v] = [v, x, u]$. Then we have $\tau^t \sigma^s S = [a, 1, b]$ for some $s, t (0 \leq s \leq 8, 0 \leq t \leq m-1)$. Since $[a, 1, b]$ belongs to P, S is in $\pi(\mathcal{P})$.

If $u, v \neq a$, then by rotating an edge $\{u, v\}$ by σ^s $(0 \leq s \leq 8)$, we get an edge $\{j, c\}, \{d, i\}, \{h, e\}$ or $\{g, f\}$. Since 2-paths $[j, x, c], [d, x, i], [h, x, e], [g, x, f]$ $(x \in V_m)$ belong to $\pi(\mathcal{Q})$, S is in $\pi(\mathcal{Q})$.

(2) Consider a 2-path $S = [u, v, x]$, $u, v \in V_{10}, u \neq v, x \in V_m$.

If $u = a$, then we have $\tau^t \sigma^s S = [a, c, 0]$ for some $s, t (0 \leq s \leq 8, 0 \leq t \leq m-1)$. Since $[a, c, 0]$ belongs to P, S is in $\pi(\mathcal{P})$.

If $v = a$, then we have $\tau^t \sigma^s S = [c, a, 1]$ for some $s, t (0 \leq s \leq 8, 0 \leq t \leq m-1)$. Since $[c, a, 1]$ belongs to P, S is in $\pi(\mathcal{P})$.

If $u, v \neq a$, then by rotating an ordered pair (u, v) by σ^s $(0 \leq s \leq 8)$, we get an ordered pair $(b, e), (e, b), (c, d), (d, c), (g, e), (e, g), (e, i)$ or (j, f). Since 2-paths $[b, e, x], [e, b, x], [c, d, x], [d, c, x], [g, e, x], [e, g, x], [e, i, x], [j, f, x]$ $(x \in V_m)$ belong to $\pi(\mathcal{P} \cup \mathcal{Q})$, S is in $\pi(\mathcal{P} \cup \mathcal{Q})$.

Therefore we have $\pi(\mathcal{P} \cup \mathcal{Q}) \supset \Pi_1 \cup \Pi_2$. This completes the proof. $\square$

(II) Construction of a set of 6-paths $\mathcal{C}$ in K_{m+10} such that $\pi(\mathcal{C}) = \Pi_3 \cup \Pi_4$.

(1) The case m is odd.

Assume m is odd and put $r = (m-1)/2$. Since $m \geq 11$, we have $r \geq 5$. Define r 6-paths as follows:

$$R_k = [d, -(k+1), k, a, -k, k+1, e] \quad (1 \leq k \leq r-1),$$
$$R_r = [e, -(r-1), -r, a, r, r-1, d].$$

Put $\mathcal{R} = T\{R_k \mid 1 \leq k \leq r\}$.

Claim 3.2 *When m is odd, $\pi(\mathcal{R}) \supset \{[x, a, y], [a, x, y] \mid x, y \in V_m, x \neq y\}$.*

Proof. For any l $(1 \leq l \leq r)$, a 2-path $[x, a, y]$ with $d(x, y) = l$ belongs to $\mathcal{R}$. Therefore $\pi(\mathcal{R}) \supset \{[x, a, y] \mid x, y \in V_m, x \neq y\}$.

For any l $(1 \leq l \leq r)$, both a 2-path $[a, x, y]$ with $y - x = l$ and a 2-path $[a, x, y]$ with $y - x = -l$ belong to $\mathcal{R}$. Therefore $\pi(\mathcal{R}) \supset \{[a, x, y] \mid x, y \in V_m, x \neq y\}$. This completes the proof. (Note that addition of elements in V_m is modulo m.) $\square$

Next define r 6-paths as follows:

$$S_k = [-k, d, k, -(k+1), b, k+1, -(k+2)] \quad (1 \leq k \leq r-2),$$
$$S_{r-1} = [-(r-1), d, r-1, -r, b, r, r-3],$$
$$S_r = [-r, d, r, (r-1), b, r-3, r-4].$$

Put $\mathcal{S} = T\{S_k \mid 1 \leq k \leq r\}$.

Claim 3.3 *When m is odd, $\pi(\mathcal{R} \cup \mathcal{S}) \supset \{[x, b, y], [x, d, y], [b, x, y], [d, x, y] \mid x, y \in V_m, x \neq y\}$.*

Proof. For any l $(1 \leq l \leq r)$, a 2-path $[x,b,y]$ with $d(x,y) = l$ and a 2-path $[x,d,y]$ with $d(x,y) = l$ belong to $\mathcal{S}$. Therefore $\pi(\mathcal{S}) \supset \{[x,b,y],[x,d,y] \mid x,y \in V_m, x \neq y\}$.

For any l $(1 \leq l \leq r)$, both a 2-path $[b,x,y]$ with $y-x = l$ and a 2-path $[b,x,y]$ with $y-x = -l$ belong to $\mathcal{S}$. Therefore $\pi(\mathcal{S}) \supset \{[b,x,y] \mid x,y \in V_m, x \neq y\}$.

For any l $(1 \leq l \leq r)$, both a 2-path $[d,x,y]$ with $y-x = l$ and a 2-path $[d,x,y]$ with $y-x = -l$ belong to $\mathcal{R} \cup \mathcal{S}$. Therefore $\pi(\mathcal{R} \cup \mathcal{S}) \supset \{[d,x,y] \mid x,y \in V_m, x \neq y\}$. This completes the proof. □

Define the vertex-permutation η in K_{m+10}: $\eta = (d\ e)(b\ c)(1\ {-1})(2\ {-2})\cdots(r\ {-r})$. Put $S'_k = \eta S_k$ $(1 \leq k \leq r)$ and $\mathcal{S}' = T\{S'_k \mid 1 \leq k \leq r\}$. Then we have the following claim.

Claim 3.4 *When m is odd,* $\pi(\mathcal{R} \cup \mathcal{S}') \supset \{[x,c,y],[x,e,y],[c,x,y],[e,x,y] \mid x,y \in V_m, x \neq y\}$.

Proof. The proof is similar to that of Claim 3.3. □

Put $\mathcal{T} = \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}'$, then we have the following claim.

Claim 3.5 *When m is odd,* $\pi(\mathcal{T}) = \{[x,u,y],[u,x,y] \mid u = a,b,c,d,e;\ x,y \in V_m, x \neq y\}$.

Proof. We have $\pi(\mathcal{T}) \supset \{[x,u,y],[u,x,y] \mid u = a,b,c,d,e;\ x,y \in V_m, x \neq y\}$ by Claims 3.2, 3.3 and 3.4. It is trivial that $\pi(\mathcal{T}) \subset \{[x,u,y],[u,x,y] \mid u = a,b,c,d,e;\ x,y \in V_m, x \neq y\}$. Therefore we have the claim. □

Define the vertex-permutation ρ in K_{m+10}: $\rho = (a\ f)(b\ g)(c\ h)(d\ i)(e\ j)$, then we have the following claim. The proof is trivial.

Claim 3.6 *When m is odd,* $\pi(\rho\mathcal{T}) = \{[x,u,y],[u,x,y] \mid u = f,g,h,i,j;\ x,y \in V_m, x \neq y\}$.

Thus we obtain Claim 3.7.

Claim 3.7 *When m is odd,* $\pi(\mathcal{T} \cup \rho\mathcal{T}) = \varPi_3 \cup \varPi_4$.

Proof. This follows from Claims 3.5 and 3.6. □

(2) The case m is even.

Assume m is even and put $r = m/2$. Since $m \geq 10$, we have $r \geq 5$. Define r 6-paths as follows:

$$X_k = \begin{cases} [e,-1,-(k+1),a,0,-(k+2),d] & (k\text{: odd, } 1 \leq k \leq r-2) \\ [d,-1,-(k+1),a,0,-(k+2),e] & (k\text{: even, } 1 \leq k \leq r-2), \end{cases}$$
$$X_{r-1} = [d,-2,-1,a,0,r-1,e],$$
$$X_r = [d,-2,0,a,r,r-2,e].$$

Put $\mathcal{X} = T\{X_k \mid 1 \leq k \leq r-1\} \cup T^*\{X_r\}$, where $T^* = \{\tau^t \mid 0 \leq t \leq r-1\}$.

Claim 3.8 *When m is even,* $\pi(\mathcal{X}) \supset \{[x,a,y],[a,x,y] \mid x,y \in V_m, x \neq y\}$.

Proof. A 2-path $[x,a,y]$ with $2 \le d(x,y) \le r-1$ belongs to $T\{X_k|1 \le k \le r-2\}$. A 2-path $[x,a,y]$ with $d(x,y)=1$ belongs to $T\{X_{r-1}\}$. A 2-path $[x,a,y]$ with $d(x,y)=r$ belongs to $T^*\{X_r\}$. Therefore $\pi(\mathcal{X}) \supset \{[x,a,y] \mid x,y \in V_m, x \ne y\}$.

A 2-path $[a,x,y]$ with $1 \le y-x \le r-2$ or $-r \le y-x \le -3$ belongs to $T\{X_k|1 \le k \le r-2\}$. A 2-path $[a,x,y]$ with $y-x=-1$ or $r-1$ belongs to $T\{X_{r-1}\}$. A 2-path $[a,x,y]$ with $y-x=-2$ belongs to $T^*\{X_r\}$. Therefore $\pi(\mathcal{X}) \supset \{[a,x,y] \mid x,y \in V_m, x \ne y\}$. This completes the proof. □

Define $r+1$ 6-paths as follows:

$$
\begin{aligned}
&Y_k = \begin{cases} [-(k+3),d,-1,-(k+1),b,0,-(k+2)] & (k\text{: odd, } 1 \le k \le r-3) \\ [-1,-(k+1),b,0,-(k+2),d,-2] & (k\text{: even, } 1 \le k \le r-3), \end{cases} \\
&Y_{r-2} = \begin{cases} [-1,-(r-1),b,0,r,d,-2] & (m \equiv 0 \pmod 4) \\ [r-2,d,-1,-(r-1),b,0,r] & (m \equiv 2 \pmod 4), \end{cases} \\
&Y_{r-1} = [-1,0,b,r,r-2,d,-2], \\
&Y_r = [-1,d,0,-(r-1),b,r,r-1], \\
&Y_{r+1} = [-2,0,b,1,r,d,r-1].
\end{aligned}
$$

Put $\mathcal{Y} = T\{Y_k \mid 1 \le k \le r-2\} \cup T^*\{Y_{r-1},Y_r,Y_{r+1}\}$.

Claim 3.9 *When m is even,* $\pi(\mathcal{X} \cup \mathcal{Y}) \supset \{[x,b,y],[x,d,y],[b,x,y],[d,x,y] \mid x,y \in V_m, x \ne y\}$.

Proof. We will give the proof only in the case $m \equiv 0 \pmod 4$. The proof in the case $m \equiv 2 \pmod 4$ is similar.

A 2-path $[x,b,y]$ with $2 \le d(x,y) \le r-1$ belongs to $T\{Y_k|1 \le k \le r-2\}$. A 2-path $[x,b,y]$ with $d(x,y)=r$ belongs to $T^*\{Y_{r-1}\}$. A 2-path $[x,b,y]$ with $d(x,y)=1$ belongs to $T^*\{Y_r,Y_{r+1}\}$. Therefore $\pi(\mathcal{Y}) \supset \{[x,b,y] \mid x,y \in V_m, x \ne y\}$.

A 2-path $[x,d,y]$ with $2 \le d(x,y) \le r-1$ belongs to $T\{Y_k|1 \le k \le r-2\}$. A 2-path $[x,d,y]$ with $d(x,y)=r$ belongs to $T^*\{Y_{r-1}\}$. A 2-path $[x,d,y]$ with $d(x,y)=1$ belongs to $T^*\{Y_r,Y_{r+1}\}$. Therefore $\pi(\mathcal{Y}) \supset \{[x,d,y] \mid x,y \in V_m, x \ne y\}$.

A 2-path $[b,x,y]$ with $1 \le y-x \le r-2$ or $-3 \ge y-x \ge -r$ belongs to $T\{Y_k|1 \le k \le r-2\}$. A 2-path $[b,x,y]$ with $y-x=-1,-2,r-1$ belongs to $T^*\{Y_{r-1},Y_r,Y_{r+1}\}$. Therefore $\pi(\mathcal{Y}) \supset \{[b,x,y] \mid x,y \in V_m, x \ne y\}$.

For any odd l $(1 \le l \le r-3)$, a 2-path $[d,x,y]$ with $y-x=-l$ belongs to $T\{Y_k|1 \le k \le r-2\}$. For any even l $(4 \le l \le r)$, a 2-path $[d,x,y]$ with $y-x=l$ belongs to $T\{Y_k|1 \le k \le r-2\}$. A 2-path $[d,x,y]$ with $y-x=-(r-1)$ belongs to $T^*\{Y_r,Y_{r+1}\}$. A 2-path $[d,x,y]$ with $y-x=2$ belongs to $T^*\{X_r,Y_{r-1}\}$. For any odd l $(1 \le l \le r-1)$, a 2-path $[d,x,y]$ with $y-x=l$ belongs to $T\{X_k|1 \le k \le r-1\}$. For any even l $(2 \le l \le r-2)$, a 2-path $[d,x,y]$ with $y-x=-l$ belongs to $T\{X_k|1 \le k \le r-1\}$. Therefore $\pi(\mathcal{X} \cup \mathcal{Y}) \supset \{[d,x,y] \mid x,y \in V_m, x \ne y\}$. This completes the proof. □

Define $r+1$ 6-paths as follows:

$$Z_k = \begin{cases} [k+3, e, 1, k+1, c, 0, k+2] & (k\text{: odd, } 1 \le k \le r-3) \\ [-(k+3), e, -1, -(k+1), c, 0, -(k+2)] & (k\text{: even, } 1 \le k \le r-3), \end{cases}$$
$$Z_{r-2} = \begin{cases} [1, e, -1, -(r-1), c, 0, r] & (m \equiv 0 \pmod 4) \\ [-(r-2), e, r, 0, c, r-1, -2] & (m \equiv 2 \pmod 4)), \end{cases}$$
$$Z_{r-1} = [r-2, e, -2, 0, c, r, -(r-1)],$$
$$Z_r = \begin{cases} [1, 0, c, -1, r, e, -(r-1)] & (m \equiv 0 \pmod 4) \\ [1, 0, c, -1, -(r-1), e, r] & (m \equiv 2 \pmod 4)), \end{cases}$$
$$Z_{r+1} = \begin{cases} [1, e, 0, r-1, c, r, r-2] & (m \equiv 0 \pmod 4) \\ [0, e, 1, r-1, c, r, r-2]. & (m \equiv 2 \pmod 4)). \end{cases}$$

Put $\mathcal{Z} = T\{Z_k \mid 1 \le k \le r-2\} \cup T^*\{Z_{r-1}, Z_r, Z_{r+1}\}$.

Claim 3.10 *When m is even, $\pi(\mathcal{X} \cup \mathcal{Z}) \supset \{[x, c, y], [x, e, y], [c, x, y], [e, x, y] \mid x, y \in V_m, x \ne y\}$.*

Proof. We will give the proof only in the case $m \equiv 0 \pmod 4$. The proof in the case $m \equiv 2 \pmod 4$ is similar.

A 2-path $[x, c, y]$ with $2 \le d(x, y) \le r-1$ belongs to $T\{Z_k | 1 \le k \le r-2\}$. A 2-path $[x, c, y]$ with $d(x, y) = r, 1$ belongs to $T^*\{Z_{r-1}, Z_r, Z_{r+1}\}$. Therefore $\pi(\mathcal{Z}) \supset \{[x, c, y] \mid x, y \in V_m, x \ne y\}$.

A 2-path $[x, e, y]$ with $2 \le d(x, y) \le r-1$ belongs to $T\{Z_k | 1 \le k \le r-2\}$. A 2-path $[x, e, y]$ with $d(x, y) = r, 1$ belongs to $T^*\{Z_{r-1}, Z_r, Z_{r+1}\}$. Therefore $\pi(\mathcal{Z}) \supset \{[x, e, y] \mid x, y \in V_m, x \ne y\}$.

For any odd l $(1 \le l \le r-3)$, a 2-path $[c, x, y]$ with $y - x = -l$ belongs to $T\{Z_k | 1 \le k \le r-2\}$. For any odd l $(3 \le l \le r-1)$, a 2-path $[c, x, y]$ with $y - x = l$ belongs to $T\{Z_k | 1 \le k \le r-2\}$. For any even l $(2 \le l \le r)$, a 2-path $[c, x, y]$ with $y - x = l$ belongs to $T\{Z_k | 1 \le k \le r-2\}$. For any even l $(4 \le l \le r-2)$, a 2-path $[c, x, y]$ with $y - x = -l$ belongs to $T\{Z_k | 1 \le k \le r-2\}$. A 2-path $[c, x, y]$ with $y - x = 1, -2, -(r-1)$ belongs to $T^*\{Z_{r-1}, Z_r, Z_{r+1}\}$. Therefore $\pi(\mathcal{Z}) \supset \{[c, x, y] \mid x, y \in V_m, x \ne y\}$.

For any odd l $(1 \le l \le r-3)$, a 2-path $[e, x, y]$ with $y - x = l$ belongs to $T\{Z_k | 1 \le k \le r-2\}$. For any even l $(2 \le l \le r-2)$, a 2-path $[e, x, y]$ with $y - x = -l$ belongs to $T\{Z_k | 1 \le k \le r-2\}$. A 2-path $[e, x, y]$ with $y - x = r-1$ belongs to $T\{Z_r, Z_{r+1}\}$. A 2-path $[e, x, y]$ with $y - x = 2$ belongs to $T\{X_r, Z_{r-1}\}$. For any odd l $(1 \le l \le r-1)$, a 2-path $[e, x, y]$ with $y - x = -l$ belongs to $T\{X_k | 1 \le k \le r-1\}$. For any even l $(4 \le l \le r)$, a 2-path $[e, x, y]$ with $y - x = l$ belongs to $T\{X_k | 1 \le k \le r-1\}$. Therefore $\pi(\mathcal{X} \cup \mathcal{Z}) \supset \{[e, x, y] \mid x, y \in V_m, x \ne y\}$. This completes the proof. □

Put $\mathcal{W} = \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$, then we have the following claim.

Claim 3.11 *When m is even, $\pi(\mathcal{W}) = \{[x, u, y], [u, x, y] \mid u = a, b, c, d, e;\ x, y \in V_m, x \ne y\}$.*

Proof. We have $\pi(\mathcal{W}) \supset \{[x, u, y], [u, x, y] \mid u = a, b, c, d, e;\ x, y \in V_m, x \ne y\}$ by Claims 3.8, 3.9 and 3.10. It is trivial that $\pi(\mathcal{W}) \subset \{[x, u, y], [u, x, y] \mid u = a, b, c, d, e;\ x, y \in V_m, x \ne y\}$. Therefore we have the claim. □

Define the vertex-permutation ρ in K_{m+10} same as before: $\rho = (a\ f)(b\ g)(c\ h)$ $(d\ i)(e\ j)$, then we have the following claim. The proof is trivial.

Claim 3.12 *When m is even,* $\pi(\rho\mathcal{W}) = \{[x,u,y],[u,x,y] \mid u = f,g,h,i,j;\ x,y \in V_m, x \neq y\}$.

Thus we obtain Claim 3.13.

Claim 3.13 *When m is even,* $\pi(\mathcal{W} \cup \rho\mathcal{W}) = \Pi_3 \cup \Pi_4$.

Proof. This follows from Claims 3.11 and 3.12. □

(III) Final step.

Let $\mathcal{U}, \mathcal{V}$ be uniform coverings of 2-paths with 6-paths in K_m, K_{10}, respectively. Then we have the following claim.

Claim 3.14
(1) *When m is odd, $\mathcal{U} \cup \mathcal{V} \cup (\mathcal{P} \cup \mathcal{Q}) \cup (\mathcal{T} \cup \rho\mathcal{T})$ is a uniform covering of 2-paths with 6-paths in K_{m+10}.*
(2) *When m is even, $\mathcal{U} \cup \mathcal{V} \cup (\mathcal{P} \cup \mathcal{Q}) \cup (\mathcal{W} \cup \rho\mathcal{W})$ is a uniform covering of 2-paths with 6-paths in K_{m+10}.*

Proof. (1) The 2-paths in K_m and the 2-paths in K_{10} are covered with $\mathcal{U} \cup \mathcal{V}$ exactly once. The 2-paths in $\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4$ are covered with $(\mathcal{P} \cup \mathcal{Q}) \cup (\mathcal{T} \cup \rho\mathcal{T})$ by Claims 3.1 and 3.7. They are covered exactly once since the 2-paths belonging to $(\mathcal{P} \cup \mathcal{Q}) \cup (\mathcal{T} \cup \rho\mathcal{T})$ are distinct. Therefore all the 2-paths in K_{m+10} are covered with $\mathcal{U} \cup \mathcal{V} \cup (\mathcal{P} \cup \mathcal{Q}) \cup (\mathcal{T} \cup \rho\mathcal{T})$ exactly once, which completes the proof of (1). The proof of (2) is similar to that of (1). □

Thus we complete the proof of Prop. 3.1. □

4 A Proof of Theorem 1.1

We will prove Theorem 1.1. Let $n \geq 7$ be an integer. Assume that there exists a uniform covering of 2-paths with 6-paths in K_n. Since the number of 2-paths in K_n is $n(n-1)(n-2)/2$ and the number of 2-paths in a 6-path is 5, $n(n-1)(n-2)/10$ must be an integer. Therefore we have $n \equiv 0,1,2 \pmod 5$.

To show the converse, we denote by A_n the following statement for an integer $n (\geq 7)$, A_n: There exists a uniform covering of 2-paths with 6-paths in K_n.

We use an induction on n satisfying $n \equiv 0,1,2 \pmod 5$. When $7 \leq n \leq 17$ and $n \equiv 0,1,2 \pmod 5$, A_n holds by Prop. 2.1.

Let $l \geq 17$ and $l \equiv 0,1,2 \pmod 5$. Assume A_k holds for all $k \leq l$ and $k \equiv 0,1,2 \pmod 5$. Let n be the smallest integer such that $n > l$ and $n \equiv 0,1,2 \pmod 5$. Since $n - 10 \leq l$ and $n - 10 \equiv 0,1,2 \pmod 5$, A_{n-10} holds by our assumption. Since $n - 10 \geq 10$, A_n holds by Prop. 3.1.

Therefore A_n holds for all $n \geq 7$ and $n \equiv 0,1,2 \pmod 5$.

This completes the proof of Theorem 1.1.

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Foldings of Regular Polygons to Convex Polyhedra I: Equilateral Triangles

Jin Akiyama and Gisaku Nakamura

Research Institute of Educational Development, Tokai University,
Shibuya, Tokyo, 151-8677 Japan
fwjb5117@mb.infoweb.ne.jp

Abstract. To *fold* a regular n-gon into a convex polyhedron is to form the polyhedron by gluing portions of the perimeter of the n-gon together, *i.e.* the n-gon is a net of the polyhedron. In this paper we identify all convex polyhedra which are foldable from an equilateral triangle.

1 Introduction

In [3, 4] the study of folding polygons to convex polyhedra was studied. In [3] they discussed the systematic construction of all polyhedra foldable from a unit square. In this paper we follow this line of research by identifying all polyhedra which are foldable from an equilateral triangle. In subsequent papers [1, 2], we identify all polyhedra which are foldable from regular n-gons ($n \geq 5$).

To *fold* a regular n-gon into a convex polyhedron means to form the polyhedron by gluing portions of the perimeter of the n-gon together, *i.e.* the n-gon is a net of the polyhedron.

2 Flat Foldings

We first discuss the degenerate case where the folding results in a flat polygon of zero volume whose shape is doubly covered by the surface of the triangle. Figure 2.1 shows a flat folding of an equilateral to a right triangle and Figure 2.2 shows a flat folding of an equilateral triangle to a rectangle.

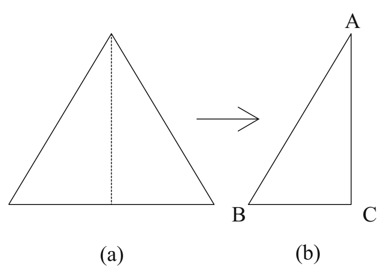

Fig. 2.1

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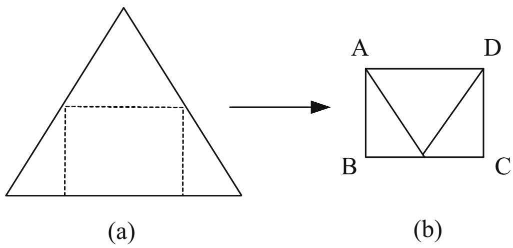


Fig. 2.2

To determine all the possible kinds of flat foldings, recall that every portion of the perimeter should be glued to another portion of the perimeter. We can start with one of the vertices of the triangle and move it along the perimeter. It is obvious that a flat folding will result if a vertex is glued to another vertex –the case shown in Figure 2.1. The only other way to obtain flat foldings is to glue a vertex to the midpoint of one of the sides of the triangle. (Figures 2.2-2.4). Otherwise, the parts that remain unfolded cannot be properly folded to doubly cover themselves.

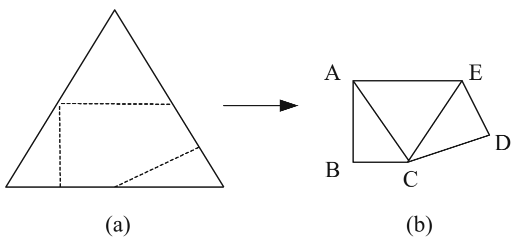


Fig. 2.3

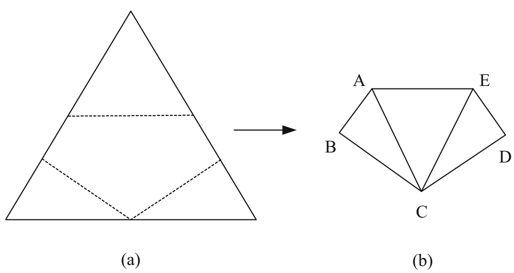


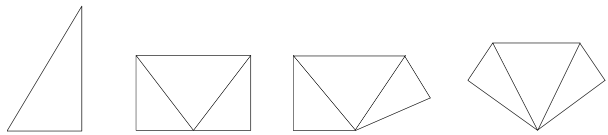
Fig. 2.4

These observations lead to the following:

Proposition 1. *The only flat foldings of an equilateral triangle are the following:*

3 Polyhedral Foldings

We now include foldings which are not flat foldings and investigate the possibilities systematically and exhaustively as follows:



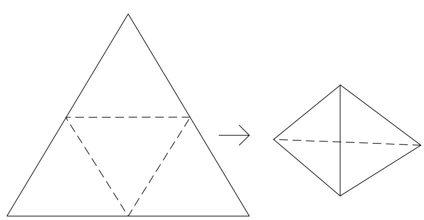
Case 1. The three vertices of the triangle converge at a point.
 a. The point of convergence is glued to a point which does not lie on the perimeter of the triangle.
 b. The point of convergence is glued to a point which lies on the perimeter of the triangle.

Case 2. Only two of the three vertices of the triangle converge at a point.
 a. The point of convergence is glued to a point which lies on the side connecting the two vertices.
 b. The point of convergence is glued to a point which lies on a side of the triangle having one of the vertices as an end point but not the other.

Case 3. None of the three vertices converge at a point.
 a. One of the vertices is glued to a point on the opposite side.
 b. None of the vertices are glued to points on the opposite side.

Case 1. The three vertices of the triangle converge at a point.

Case 1a.
There is only one way to achieve this folding, namely, to fold the original triangle along the three dotted lines in Figure 3.1(a). This yields a regular tetrahedron (Figure 3.1(b)).



(a) (b)

Fig. 3.1

Case 1b.
The sum of the angles at the vertices is π and the angle formed by the side at the point is also π, so the total sum of the angles is 2π, hence this point of

convergence cannot be one of vertices of a convex polyhedron. Consequently, this point must either lie on an edge of the resulting polyhedron or else be an interior point of one of the faces of the polyhedron. The second alternative results in a flat folding. Thus, it remains to consider the first alternative.

The flat folded rectangle in Figure 2.2(b) is the degenerate analog of the case under consideration. It helps to begin the discussion with this rectangle. Label the vertices of the rectangle ABCD and let E be the point of convergence of the three vertices of the triangle (Figure 3.2). Split edge BC of the flat folded rectangle and pull the two sides apart to form a rolling belt as shown in Figure 3.2. Let F denote the point on the rolling belt diametrically opposite the point E. Let G be another point on the rolling belt. Bend the figure so that E is glued to G. This will induce a folding which yields a tetrahedron.

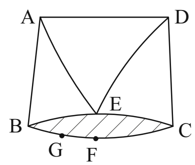


Fig. 3.2

To see how this tetrahedron is formed, consider Figure 3.3(a). The dotted lines on the triangle indicate where the folds should be placed to achieve the desired result. Fixing the location of G will automatically determine the location of the folds. The two angles labeled α are equal, otherwise a concave hexahedron will result. (To visualize this hexahedron, think of gluing two tetrahedra base-to-base.)

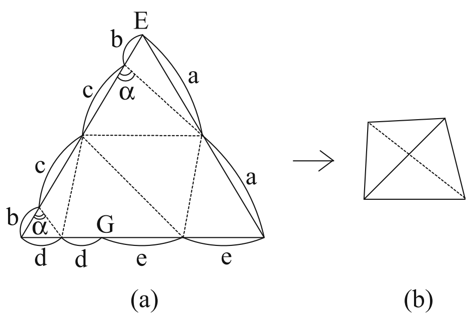


Fig. 3.3

Since the location of the point G can be chosen arbitrarily on the perimeter of the rolling belt, a continuum of tetrahedra is obtained as the location of G moves along the rolling belt. Figure 3.4 illustrates the progression. The flat folded rectangle is at the bottom while the regular tetrahedron is at the top. As the location of G moves along the rolling belt, the shape of the tetrahedron changes continuously. This completes the discussion of Case 1b.

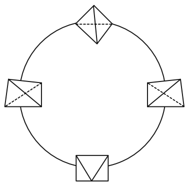


Fig. 3.4

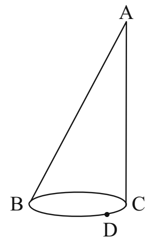


Fig. 3.5

Case 2. Only two of the three vertices of the triangle converge at a point.

In Case 2, the point of convergence of the two vertices will surely be glued to a point on a side of the triangle since the sum of the angles at the point of convergence is less than π. Furthermore, when the point of convergence has been glued the total sum of the angles at the common point is strictly less than 2π, hence this point will be a vertex of the resulting polyhedron.

Case 2a.
We again start with the degenerate analog of this case, the flat folded right triangle in Figure 2.1(b). Split the flat folding at BC and pull the two sides apart to form a rolling belt as shown in Figure 3.5. Vertex B has to be glued to a point D on the circumference of the rolling belt to conform with the conditions of this case. A thin tetrahedron will result. Figure 3.6 illustrates the procedure. The dotted lines in Figure 3.6(a) indicate the locations of the folds. Again, the two angles labeled α are equal, otherwise, a concave hexahedron results.

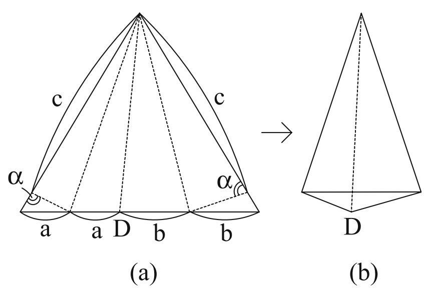


Fig. 3.6

We note that once the location of the point D is fixed, then the location of the folds will be automatically determined. As the location of point D moves along the circumference of the rolling belt, a continuum of tetrahedra is obtained. Figure 3.7 illustrates the progression. The flat folding is at the top, while a thin tetrahedron whose bottom face is an isosceles triangle is at the bottom.

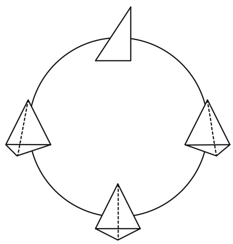


Fig. 3.7

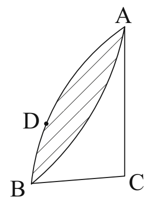


Fig. 3.8

Case 2b.
We start with the degenerate analog of this case which is again the flat folded right triangle of Figure 2.1(b). This time we split the hypotenuse AB and pull apart the sides to form a rolling belt (Figure 3.8). Vertex A has to be glued to a point D on the circumference of the rolling belt to conform with the conditions of this case. Note that if A is glued to B, then we revert back to the regular tetrahedron of Case 1a. Also, if A is glued to a point D which is a distance $\frac{1}{4}$ or $\frac{3}{4}$ away from A along the circumference of the rolling belt, then we obtain the flat folded pentagon in Figure 2.3(b).

Allowing the position of D to vary between A and a point $\frac{1}{4}$ away from A on the rolling belt, then gluing A to D, results in a convex four-sided cone with a rectangular base. Figure 3.9 illustrates the process. As D traverses this portion of the rolling belt, a continuum of four-sided cones is obtained.

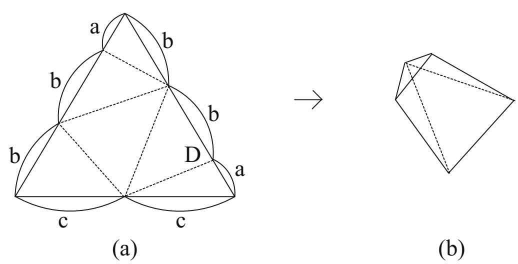


Fig. 3.9

We continue by allowing D to vary between a point $\frac{1}{4}$ away from A and the point half-way away from A on the rolling belt, *i.e.* the point B. Then gluing A to D results in a convex hexahedron as shown in Figure 3.11. Figure 3.10 illustrates the process. The folding must be such that the two angles labeled

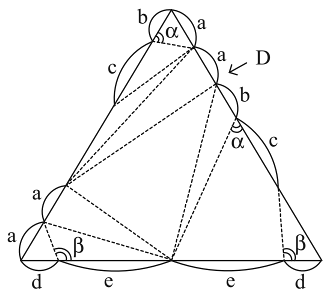


Fig. 3.10

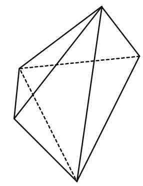


Fig. 3.11

α are equal, so also with the two angles labeled β. If these conditions are not satisfied, a concave decahedron will result.

As the location of the point D varies, we obtain a continuum of four-sided cones. The progression is shown in Figure 3.12. We start at the lowest point with the degenerate case, which is represented by the flat-folded right triangle, and move (say counterclockwise) to the point $\frac{1}{4}$ away from A on the rolling belt which is represented by the flat folded pentagon. We then move to the point $\frac{1}{2}$ away from A on the rolling belt, which is represented by the regular tetrahedron and so on.

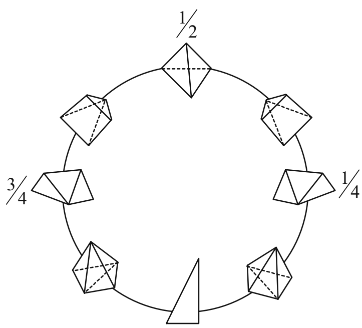


Fig. 3.12

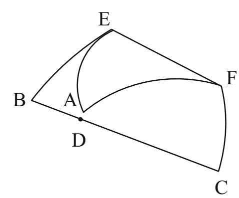


Fig. 3.13

Case 3. None of the three vertices converge at a point.

In Case 3, it is easy to see that the gluing will not result in a tetrahedron unless each vertex is glued to some point on the perimeter of the original triangle.

Case 3a.

The situation described in this case is illustrated in Figure 3.13. The vertex A is glued to a point D located on the side of the triangle opposite A. The perimeter of the original triangle is now divided two parts ABE and ACF. Assuming that the lengths of these two parts are equal so that a gluing is possible, we note that gluing these parts together will not result in a convex polyhedron but a torus. Hence it is necessary to glue these two portions separately.

Note that the sum of the angles coming together at D is less than 2π, hence this point will become a vertex of the resulting polyhedron. Once the location of the point D on the edge BC is determined, the shape of the resulting polyhedron will be determined uniquely. Figure 3.14 illustrates the process. The two angles labeled α in the diagram must be equal otherwise a concave octahedron will result. (This octahedron has the shape of two tetrahedra glued together base-to-base and its surface consists of eight triangles.)

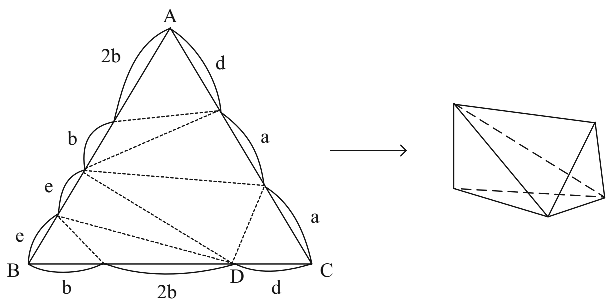


Fig. 3.14

When the point D coincides with either B or C, then the flat folded triangle of Figure 2.1(b) results. When D coincides the midpoint of BC, then the flat folded pentagon of Figure 2.4(b) results. As D varies between these specified points, various hexahedra result. The progression is illustrated in Figure 3.15.

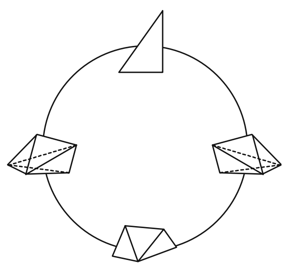


Fig. 3.15

Case 3b.
The possibilities in this case are confined to the two shown in Figure 3.16. No convex polyhedron will result from either of these two possibilities. To see why this is so, we focus on vertex A (Figure 3.17). If we glue A to point D, then D must lie on either AB or AC or else AF and AG must converge at this point. In the first case, the sum of the vertex angles coming together at the point will

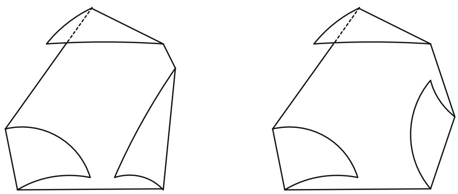


Fig. 3.16

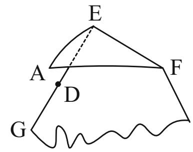


Fig. 3.17

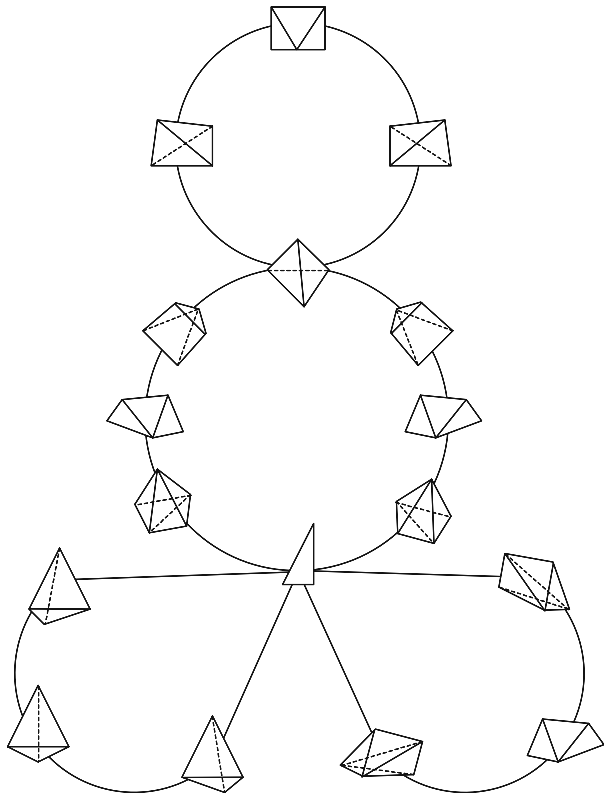


Fig. 3.18

exceed 2π, which is impossible. On the other hand, if AF and AG converge at point D, then the same kind of thing must happen to the other two vertices as well. However, if it is possible to construct a convex polyhedron in this way, then the line segment obtained by gluing together the sides AF and AG would become part of an edge of the resulting polyhedron, and hence another vertex of this polyhedron should be located at the other end of this edge. It is clear that this other vertex must come from either vertex B or C of the original triangle;

but, this means that if, for example, the vertex of the resulting polyhedron coming from the original vertex A matches the vertex coming from vertex B, then there will be no vertex of the resulting polyhedron which matches the vertex coming from the vertex C. Thus, it is impossible for a convex polyhedron to be constructed.

The preceding discussion comprises the proof of the following theorem.

Theorem 1. *An equilateral triangle can be folded into polyhedra having from three to five vertices, falling into six distinct combinatorial classes: hexahedra, pentahedra, tetrahedra, flat folded right triangles, rectangles and pentagons.*

The route diagram in Figure 3.18 shows all possible polyhedra that are constructible by folding an equilateral triangle and illustrates how the infinitely many continuously changing polyhedra will arise.

Acknowledgements

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Maximum Induced Matchings of Random Regular Graphs*

Hilda Assiyatun

Department of Mathematics, Institut Teknologi Bandung,
Bandung 40124, Indonesia
hilda@dns.math.itb.ac.id

Abstract. An *induced matching* of a graph $G = (V, E)$ is a matching $\mathcal{M}$ such that no two edges of $\mathcal{M}$ are joined by an edge of $E \backslash \mathcal{M}$. In general, the problem of finding a maximum induced matching of a graph is known to be NP-hard. In random d-regular graphs, the problem of finding a maximum induced matching has been studied for $d \in \{3, 4, \ldots, 10\}$. This was due to Duckworth et al.(2002) where they gave the asymptotically almost sure lower bounds and upper bonds on the size of maximum induced matchings in such graphs. The asymptotically almost sure lower bounds were achieved by analysing a degree-greedy algorithm using the differential equation method, whilst the asymptotically almost sure upper bounds were obtained by a direct expectation argument. In this paper, using the *small subgraph conditioning method*, we will show the asymptotically almost sure existence of an induced matching of certain size in random d-regular graphs, for $d \in \{3, 4, 5\}$. This result improves the known asymptotically almost sure lower bound obtained by Duckworth et al.(2002).

1 Introduction

A matching of a graph, $G = (V, E)$, is a set of edges, $M \subseteq E(G)$, such that no two edges of M share a common end-point. An *induced matching*, $\mathcal{M}$, of a graph, $G = (V, E)$, is a matching such that no two edges of $\mathcal{M}$ are joined by an edge of $E(G) \backslash \mathcal{M}$.

The problem of finding a maximum induced matching of a graph is known to be NP-hard in general, even in bipartite graphs and in planar cubic graphs (see [5, 7, 15]). The problem has been shown to be solvable in polynomial time for several classes of graphs, such as for chordal graphs and interval graphs [5] and for circular-arc graphs [10]. More recently, Golumbic and Lewenstein [9] showed that in an interval graph finding a maximum induced matching of a graph can be computed in linear time.

As we are concerned with random regular graphs, we need some notation. We use the notation **P** (probability), **E** (expectation) and **Var** (variance). We

* The research was carried out while the author was in the Department of Mathematics & Statistics, the University of Melbourne, Australia.

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say an event Y_n occurs a.a.s. (asymptotically almost surely) if $\mathbf{P}Y_n \to 1$ as n goes to infinity.

In random d-regular graphs, the problem of finding a maximum induced matching has been studied for $d \in \{3, 4, \ldots, 10\}$. This was due to Duckworth et al. [3] where they gave the asymptotically almost sure lower bounds and upper bonds on the size of maximum induced matchings in such graphs. The asymptotically almost sure lower bounds were achieved by analysing a degree-greedy algorithm using the differential equation method, whilst the asymptotically almost sure upper bounds were obtained by a direct expectation argument. In [6] Duckworth et al. considered only the case of $d = 3$. The a.a. sure lower bound obtained in this paper is corrected in [3].

In this paper, using the small subgraph conditioning method, we will show the almost sure existence of an induced matching of certain size in random d-regular graphs, for $d \in \{3, 4, 5\}$. For these values of d the size of induced matchings obtained by the method improve the known a.a. sure lower bound in [3].

The small subgraph conditioning method, which is a technique of analysing variance, is due to Robinson and Wormald [14, 13] where they proved the a.a. sure hamiltonicity of random d-regular graphs. The method also can be used to prove the a.a. sure existence, and the asymptotic distribution, of properties of random regular graphs.

Janson [12] and Robalewska [11] successfully used this method to determine the a.a. sure existence, and the asymptotic distribution, of the number of 1-regular and 2-regular spanning subgraphs in random regular graphs. Garmo [8] applied the method to the number of long cycles in random d-regular graphs. In more recent work, Assiyatun and Duckworth [2] have used the method to study maximal matchings in such graphs.

Let $\mathcal{G}_{n,d}$ denote the probability space of d-regular graphs on n vertices. In asymptotic statements about properties of $\mathcal{G}_{n,d}$ we restrict n to even integers when d is odd.

The main results obtained in this paper are presented in the following theorem and corollary.

Theorem 1. *Let $3 \le d \le 5$ and $0 < \tau < d/(4d-2)$ be fixed. Define $Y'_{d,\rho} = Y'_{d,\rho}(n)$ to be the number of induced matchings of size $p = \lfloor \tau n \rfloor$ in $G \in \mathcal{G}_{n,d}$, and define ρ by $\rho n = p$. If $\tau = \beta_1(d)$, with $\beta_1(d)$ as given in Table 1, then $G \in \mathcal{G}_{n,d}$ a.a.s. has an induced matching of size ρn. Moreover,*

$$\frac{Y'_{d,\rho}}{\mathbf{E}Y'_{d,\rho}} \xrightarrow{d} W = \Pi_{k\ge 3}^{\infty} (1+\delta_k)^{Z_k} e^{-\lambda_k \delta_k} \ \text{as } n \to \infty,$$

where Z_k are independent Poisson variables with $\mathbf{E}Z_k = \lambda_k$ for $k \ge 3$, with $\lambda_k = \frac{(d-1)^k}{2k}$ and $\delta_k = 2\,\mathrm{Re}(\mu^k)$ where $\mu = \frac{-(d-1)\rho + \sqrt{(d^2+2d+1)\rho^2 - 2d\rho}}{d(1-2\rho)}$.

Note that $\rho \to \tau$ as n goes to infinity.

Corollary 1. *For a fixed $3 \le d \le 5$ define $\mathcal{M}$ to be a maximum induced matching in $G \in \mathcal{G}_{n,d}$ Then a.a.s. $\beta_1(d)n < |\mathcal{M}| < \beta_2(d)n$, with $\beta_1(d)$ and $\beta_2(d)$ are given in Table 1.*

Table 1. The a.a. sure bounds on $|\mathcal{M}|/n$

d	β_1	β_2	β_1'
3	0.2761	0.2821	0.26645
4	0.2431	0.2500	0.22953
5	0.2201	0.2270	0.20465

β_1: the a.a. sure lower bound obtained in Theorem 1

β_2: the a.a. sure upper bound obtained in Theorem 2

β_1': the a.s. sure lower bound obtained in [3]

Instead of working directly with $\mathcal{G}_{n,d}$, we will use the *pairing model* which is originally due to Bender and Canfield. Bollobás gave a simplified form of this model which can be described in as follows.

Let $V = \bigcup_{i=1}^{n} V_i$ be a fixed set of dn points, where $|V_i| = d$, for every i. A perfect matching of the points of V into $dn/2$ pairs is called a *pairing.* A pairing P corresponds to a d-regular pseudograph $G(P)$ in which each V_i is regarded as a vertex and each pair is an edge. We use $\mathcal{P}_{n,d}$ to denote the probability space of all pairings.

For two sequences a_n and b_n, we denote $a_n \sim b_n$ if the ratio $\frac{a_n}{b_n}$ tends to 1 as n goes to infinity. We denote the falling factorial $n(n-1)\cdots(n-m+1)$ by $[n]_m$. We define $N(2m) = \frac{(2m)!}{2^m m!}$, which is the number of perfect matchings of $2m$ points.

Let $d \geq 3$ and $0 < \tau < d/(4d-2)$ be fixed. In what follows, we define $Y_{d,\rho} = Y_{d,\rho}(n)$ to be the number of induced matchings of size $p = \lfloor \tau n \rfloor = \rho n$ in a random d-regular pseudograph coming from a pairing $P \in \mathcal{P}_{n,d}$. The proof of Theorem 1 uses the small subgraph conditioning method. This method may be extracted in one theorem (see Janson [12–Theorem 10] or Wormald [16–Theorem 4.1]). The main work in this method is the computation of the expectation and variance of $Y_{d,\rho}$ (which we present in Section 2), and the computation of the conditional expectation of $Y_{d,\rho}$ based on the distribution of short cycles (which we present in Section 3). The proofs of Theorem 1 and Corollary 1 are presented in Section 4.

2 Expectation and Variance of $Y_{d,\rho}$

Duckworth et al. [6–Theorem 2] proved the next theorem for $d = 3$.

Theorem 2.

$$\mathbf{E}Y_{d,\rho} \sim \sqrt{\frac{d}{2\pi n\rho(d-(4d-2)\rho)}}\,\psi(d,\rho)^n$$

with

$$\psi(d,\rho) = \frac{(1-2\rho)^{(d-1)(1-2\rho)}d^{d/2-(2d-2)\rho}}{2^\rho \rho^\rho (d-(4d-2)\rho)^{d/2-(2d-1)\rho}}.$$

Moreover, there exists a unique $\beta_2(d) \in \left(0, \frac{d}{4d-2}\right)$ *such that*

$$\psi(d,\rho) \begin{cases} > 1 & \text{if } 0 < \rho < \beta_2(d) \\ \leq 1 & \text{if } \beta_2(d) \leq \rho < \frac{d}{4d-2} \end{cases}.$$

Proof. The number of ways to choose a set of $p = \rho n$ independent edges, M, together with the points used is

$$\frac{n! d^{2p}}{2^p p! (n-2p)!}. \tag{1}$$

Since M is induced then all the remaining edges incident with the vertices that are the end-points of the matching edges (there are $2(d-1)p$ such edges) must be incident with the other $n-2p$ vertices. Hence, the number of ways to assign these edges is

$$[dn - 2dp]_{2(d-1)p} = \frac{(dn-2dp)!}{(dn-(4d-2)p)!}.$$

As the number of unused points in the pairing is $dn - (4d-2)p$, the pairing can be completed in $N(dn-(4d-2)p)$ ways. Multiplying these we obtain the number of pairings containing an induced matching of size p,

$$\frac{n!(dn-2dp)! d^{2p}}{2^p p!(n-2p)!(dn-(4d-2)p)!} N(dn-(4d-2)p). \tag{2}$$

Dividing this by $N(dn)$, the number of total pairings in $\mathcal{P}_{n,d}$, we obtain

$$\begin{aligned} \mathbf{E} Y_{d,\rho} &= \frac{n! d^{2p}}{2^p p!(n-2p)!} \times \frac{(dn-2dp)!}{(dn-(4d-2)p)!} \times \frac{N(dn-(4d-2)p)}{N(dn)} \\ &= \frac{n!(dn-2dp)!(dn/2)! d^{2p} 2^{(2d-2)p}}{p!(n-2p)!(dn/2-(2d-1)p)!(dn)!}. \end{aligned}$$

Applying Stirling's formula to the above equation, we obtain the desired expectation.

To show the last part of the theorem, we note that for all $d \geq 3$, $\psi(d,0) = 1$ and $\psi(d, d/(4d-2)) = (d-1)^{(d-1)^2/(2d-1)}(2d-1)^{1-d/2} < 1$. Consider

$$\begin{aligned} \frac{\partial}{\partial \rho} \log \psi(d,\rho) &= -2(d-1)\log(1-2\rho) - \log \rho + (2d-1)\log(d-(4d-2)\rho) \\ &\quad -2(d-1)\log d - \log 2. \end{aligned}$$

It is easy to see that the limit of the above partial derivative is positive infinite if $\rho \to 0^+$, and is negative infinite if $\rho \to (d/(4d-2))^-$. This and the fact that

$$\begin{aligned} \frac{\partial^2}{\partial \rho^2} \log \psi(d,\rho) &= \frac{4(d-1)}{1-2\rho} - \frac{1}{\rho} - \frac{(2d-1)(4d-2)}{d-(4d-2)\rho} \\ &= -\frac{(2d-4)(2d-1)\rho + d}{\rho(1-2\rho)(d-(4d-2)\rho)} < 0 \end{aligned}$$

for $\rho \in \left(0, \frac{d}{4d-2}\right)$, imply the existence of a unique maximum in the open interval. The existence and the uniqueness of $\beta_2(d)$ is now straightforward. ∎

Theorem 3. *If $3 \leq d \leq 5$ is fixed and $\tau = \beta_1(d)$ as in Table 1, then*

$$\mathbf{Var}Y_{d,\rho} \sim (C(d,\rho) - 1)(\mathbf{E}Y_{d,\rho})^2$$

where

$$C(d,\rho) = \frac{d^2(1-2\rho)^2}{(d-(4d-2)\rho)\sqrt{(-4d^3+12d^2-12d+8)\rho^2 - 4d\rho + d^2}}.$$

Proof. To obtain the variance of $Y_{d,\rho}$ we first calculate $\mathbf{E}Y_{d,\rho}(Y_{d,\rho}-1)$. Let M_1 and M_2 be two induced matchings, each of size $p = \rho n$. Let $V(M_i)$ be the vertex set of M_i, for $i \in \{1,2\}$.

Given M_1, suppose that M_2 has x' edges overlapping with M_1 and y' edges sharing one end-point with M_1 (see Figure 1). Note that the remaining $p-x'-y'$ independent edges in each M_i must be vertex disjoint. Thus the number of ways to choose the intersection is

$$\binom{p}{x'}\binom{p-x'}{y'}2^{y'}. \tag{3}$$

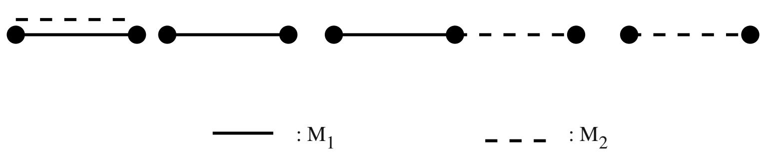

Fig. 1. Two intersecting induced matchings

In M_2 the completion of the y' edges requires y' vertices chosen from $n-2p$ vertices available, whilst the creation of the vertex-disjoint $p-x'-y'$ edges requires $2(p-x'-y')$ vertices from the leftover $n-2p-y'$ vertices. Thus given M_1 and the intersection, the number of ways to complete M_2, together with the points used, is

$$\frac{(n-2p)!(d-1)^{y'}d^{y'}}{(n-2p-y')!} \times \frac{(n-2p-y')!d^{2(p-x'-y')}}{2^{p-x'-y'}(p-x'-y')!(n-4p+2x'+y')!}. \tag{4}$$

If there are z' edges lying from $V(M_1)\backslash V(M_2)$ to $V(M_2)\backslash V(M_1)$ then the pairs of points in the pairing corresponding to these edges can be selected in

$$\binom{(d-1)(2p-2x'-y')}{z'}^2 z'! \tag{5}$$

ways.

At this stage we observe that the number of unused points in $V(M_1) \cup V(M_2)$ is $(4d-4)p-(2d-2)x'-dy'-2z'$. These points must be adjacent to the remaining $n-4p+2x'+y'$ vertices (each still has d points). Hence, the number of ways to assign these points is

$$\begin{aligned} &[d(n-4p+2x'+y')]_{(4d-4)p-(2d-2)x'-dy'-2z'} \\ &= \frac{(d(n-4p+2x'+y'))!}{(dn-(8d-4)p+(4d-2)x'+2dy'+2z')!}. \end{aligned} \tag{6}$$

The number of unused points in the pairing is $dn-(8d-4)p+(4d-2)x'+2dy'+2z'$. Therefore the number of ways to complete the pairing is

$$N(dn-(8d-4)p+(4d-2)x'+2dy'+2z'). \tag{7}$$

By multiplying (1)–(7) together and then dividing by $N(dn)$ we have

$$\begin{aligned} \mathbf{E}Y_{d,\rho}(Y_{d,\rho}-1) = {} & \frac{n!(dn/2)!d^{4p}2^{(4d-4)p}}{(dn)!} \\ & \times \sum_{R'} \left[\frac{((d-1)(2p-2x'-y'))!^2}{((d-1)(2p-2x'-y')-z')!^2(p-x'-y')!^2} \right. \\ & \times \frac{(d(n-4p+2x'+y'))!}{(dn/2-(4d-2)p+(2d-1)x'+dy'+z')!} \\ & \left. \times \frac{(d-1)^{y'}d^{-2x'-y'}2^{-(2d-2)x'-(d-2)y'-z'}}{(n-4p+2x'+y')!x'!y'!z'!} \right], \end{aligned}$$

where

$$\begin{aligned} R' = \{(x',y',z') : x',y',z' \ge 0,\ (d-1)(2p-2x'-y')-z' \ge 0,\ p-x'-y' \ge 0, \\ dn/2-(4d-2)p+(2d-1)x'+dy'+z' \ge 0\}. \end{aligned}$$

Set $x = \frac{x'}{n}, y = \frac{y'}{n}$, and $z = \frac{z'}{n}$. We will assume that all arguments in the factorial above go to infinity with n. Thus Stirling's formula gives

$$\mathbf{E}Y_{d,\rho}(Y_{d,\rho}-1) \sim \frac{1}{8(\pi n)^{5/2}} \left(d^{4\rho-d/2}2^{(4d-4)\rho-d/2}\right)^n \sum_R \alpha(x,y,z)F(x,y,z)^n \tag{8}$$

where

$$\begin{aligned} R = \{(x,y,z) : x,y,z \ge 0,\ (d-1)(2\rho-2x-y)-z \ge 0,\ \rho-x-y \ge 0, \\ d/2-(4d-2)\rho+(2d-1)x+dy+z \ge 0\}, \end{aligned} \tag{9}$$

$$\begin{aligned} F(x,y,z) = {} & \frac{f((d-1)(2\rho-2x-y))^2}{f((d-1)(2\rho-2x-y)-z)^2 f(\rho-x-y)^2} \\ & \times \frac{f(d(1-4\rho+2x+y))}{f(d/2-(4d-2)\rho+(2d-1)x+dy+z)} \\ & \times \frac{(d-1)^y d^{-2x-y}2^{-(2d-2)x-(d-2)y-z}}{f(1-4\rho+2x+y)f(x)f(y)f(z)} \end{aligned} \tag{10}$$

with $f(x) = x^x$ and

$$\alpha(x,y,z) = \left(\frac{d(d-1)^2}{(\rho-x-y)^2((d-1)(2\rho-2x-y)-z)^2}\right)^{1/2} \times \left(\frac{(2\rho-2x-y)^2}{(d/2-(4d-2)\rho+(2d-1)x+dy+z)xyz}\right)^{1/2}.$$

Next we will find the main contribution to the sum which will come from the maximum of F. We will show that for a specific value of ρ,

$$\mathbf{x}_{\max} = \left(\frac{2\rho^2}{d}, \frac{4(d-1)\rho^2}{d}, \frac{4(d-1)^2\rho^2}{d}\right) \tag{11}$$

with

$$F(\mathbf{x}_{\max}) = \frac{(1-2\rho)^{2(d-1)(1-2\rho)} d^{3d/2-4d\rho} 2^{d/2-(4d-2)\rho}}{\rho^{2\rho}(d-(4d-2)\rho)^{d-(4d-2)\rho}}$$

is the local maximum point of greatest value of F in the interior of R.

We look for all critical points of F in the interior of R. Setting the partial derivatives of $\log F$, with respect to x, y and z, equal to 0 we obtain three equations:

$$\begin{aligned}
&2d^{2d-2}(1-4\rho+2x+y)^{2d-2}(\rho-x-y)^2((d-1)(2\rho-2x-y)-z)^{4d-4} \\
&-(d-1)^{4d-4}(d-(8d-4)\rho+(4d-2)x+2dy+2z)^{2d-1}(2\rho-2x-y)^{4d-4}x = 0 \\
&4d^{d-1}(1-4\rho+2x+y)^{d-1}(\rho-x-y)^2((d-1)(2\rho-2x-y)-z)^{2d-2} \\
&-(d-1)^{2d-3}(d-(8d-4)\rho+(4d-2)x+2dy+2z)^d(2\rho-2x-y)^{2d-2}y = 0 \\
&((d-1)(2\rho-2x-y)-z)^2-(d-(8d-4)\rho+(4d-2)x+2dy+2z)z = 0.
\end{aligned}$$

Substituting the second equation into the first and then the third equation into the second consecutively we have:

$$\begin{aligned}
&(d-(8d-4)\rho+(4d-2)x+2dy+2z)y^2-8(d-1)^2(\rho-x-y)^2x = 0 \\
&4d^{d-1}(1-4\rho+2x+y)^{d-1}(\rho-x-y)^2z^{d-1} \\
&-(d-1)^{2d-3}(d-(8d-4)\rho+(4d-2)x+2dy+2z)(2\rho-2x-y)^{2d-2}y = 0 \\
&((d-1)(2\rho-2x-y)-z)^2-(d-(8d-4)\rho+(4d-2)x+2dy+2z)z = 0.
\end{aligned}$$

The above system can be slightly simplified by substituting the first equation into the second. This yields:

$$(d-(8d-4)\rho+(4d-2)x+2dy+2z)y^2-8(d-1)^2(\rho-x-y)^2x = 0 \tag{12}$$

$$d^{d-1}(1-4\rho+2x+y)^{d-1}yz^{d-1}-2(d-1)^{2d-1}(2\rho-2x-y)^{2d-2}x = 0 \tag{13}$$

$$((d-1)(2\rho-2x-y)-z)^2 -(d-(8d-4)\rho+(4d-2)x+2dy+2z)z = 0. \tag{14}$$

It is simple to verify that $\mathbf{x}_{\max}$ satisfies the above system. Due to the complexity of the function involved, in finding all interior critical points of F we restrict ourselves to a particular value of ρ. Let f_i, $i = 1, 2, 3$ be the functions on the left hand-side of (12), (13) and (14) respectively. Fix $d = d_0$ with $d_0 = 3, 4$ and 5. Solve f_1 for z since z is linear in f_1, and denote the result by f_z. Substituting $z = f_z$ into f_2 and f_3 results in

$$\frac{f_{2,z}(\rho - x - y)}{y} = 0 \text{ and } \frac{f_{3,z}}{y} = 0,$$

where $f_{2,z}$ and $f_{3,z}$ are certain functions of x, y and ρ. Since we are concerned only with the roots lying in the interior of R, it suffices to look for the common roots of $f_{2,z}$ and $f_{3,z}$. Taking the resultant of $f_{2,z}$ and $f_{3,z}$ with respect to x, yields:

$$Cy(d_0 y - 4(d_0 - 1)\rho^2)PQ,$$

where C is a constant,

$$\begin{aligned}
P(3, \rho, y) &= 5y - 18\rho + 3,\\
P(4, \rho, y) &= 5y - 16\rho + 2,\\
P(5, \rho, y) &= 17y - 50\rho + 5,
\end{aligned}$$

and Q is a certain (large) polynomial in y and ρ. Therefore, the possible roots come from zeros of $d_0 y - 4(d_0 - 1)\rho^2$ or P or Q. We ignore the root $y = 0$ which lies on the boundary of R. Due to page limitations, the functions $f_{2,z}$, $f_{3,z}$, Q, and the constant C cannot be displayed here. Interested readers may find them in [1].

Now fix $\rho = \beta_1 = \beta_1(d_0)$. Substituting back $y = 4(d_0 - 1)\beta_1^2/d_0$ into $f_{2,z}$ and $f_{3,z}$ yields $x = (2\beta^2)/d_0$ only. From f_z, these values of y and x uniquely result in $\mathbf{x}_{\max}$ as in (11). Investigating the Hessian of $\log F$ at $\mathbf{x}_{\max}$ we obtain a stronger result on the behaviour of this point, that $\mathbf{x}_{\max}$ is a local maximum of F for $0 < \rho < d_0/(4d_0 - 2)$.

On the other hand, we find that the root coming from P, which we denote by $y_1(d_0) = y_1(d_0, \beta)$, does not give a rise to a feasible root since it is larger than β_1. More precisely, we obtain $y_1(3) \approx 0.3940$, $y_1(4) \approx 0.3780$, and $y_1(5) \approx 0.3532$.

Letting $\mathrm{int}(R)$ be the interior of R, we look for $Y_Q = \{y_Q | y_Q \in \mathrm{int}(R), Q(\beta_1, y_Q) = 0\}$. Using back substitution, we find, if any, the common root, x_Q, of $f_{2,z}(\beta_1, y_Q)$ and $f_{3,z}(\beta_1, y_Q)$ for $y_Q \in Y_Q$. Then we obtain, if any, a unique $z_Q \in \mathrm{int}(R)$ by substituting $\{\rho = \beta_1, x = x_Q, y = y_Q\}$ into f_z. From this procedure we obtain two other critical points in the interior of R, namely $\mathbf{x}_1$ and $\mathbf{x}_2$. These critical points along with their nature (determined from the Hessian of $\log F$ at the corresponding point for $\rho = \beta_1$) and values of F are given in Table 2. The given β_1 is the largest ρ (to 4 decimal places) such that F attains its maximum on $\mathbf{x}_{\max}$. The assertion follows.

To show that F does not have any local maximum on the boundary of R, denoted by ∂R, we use [1–Lemma 3.3]. This lemma is a generalisation of the approach used by Garmo [8]. The main idea is to show that for every point $\mathbf{x}_0 \in \partial R$ there exists a unit vector $\mathbf{v}_0$ such that the directional derivative of F at $\mathbf{x}_0$ in the direction of $\mathbf{v}_0$ is positive infinite. This implies that F cannot reach any maximum on ∂R.

Table 2. The stationary points of F

d	$\mathbf{x}$	$F(\mathbf{x})$	Hessian $\log(F(\mathbf{x}))$
	$\mathbf{x}_{\max} \approx (0.050821, 0.203283, 0.406567)$	1.044423	negative definite
3	$\mathbf{x}_1 \approx (0.133442, 0.130621, 0.220060)$	1.040710	indefinite
	$\mathbf{x}_2 \approx (0.214568, 0.055452, 0.063072)$	1.044421	negative definite
	$\mathbf{x}_{\max} \approx (0.029549, 0.177293, 0.531878)$	2.449157	negative definite
4	$\mathbf{x}_1 \approx (0.103676, 0.121879, 0.267492)$	2.435512	indefinite
	$\mathbf{x}_2 \approx (0.183460, 0.051793, 0.063600)$	2.448734	negative definite
	$\mathbf{x}_{\max} \approx (0.019378, 0.155021, 0.620083)$	7.403373	negative definite
5	$\mathbf{x}_1 \approx (0.086323, 0.113488, 0.294027)$	7.354085	indefinite
	$\mathbf{x}_2 \approx (0.163761, 0.048027, 0.061733)$	7.400908	negative definite

Now that we have proved that F reaches its maximum at $\mathbf{x}_{\max}$, we can approximate the sum in (8) within a small closed ball around $\mathbf{x}_{\max}$, $B = B(\mathbf{x}_{\max}, n^{-2/5})$. The Taylor expansion of F at $\mathbf{x}_{\max}$ is

$$F(\mathbf{x})^n = F(\mathbf{x}_{\max})^n \times \exp\left(-n\left(as_1^2+bs_2^2+cs_3^2+es_1s_2+fs_1s_3+gs_2s_3\right)+O\left(n^{-1/5}\right)\right)$$

where $s_1 = x - 2\rho^2/d, \quad s_2 = y - 4(d-1)\rho^2/d, \quad s_3 = z - 4(d-1)^2\rho^2/d$, and

$$a = -h_{11}/2,\ b = -h_{22}/2,\ c = -h_{33}/2,\ e = -h_{12},\ f = -h_{13},\ g = -h_{23},$$

where h_{ij}, the $(ij)^{th}$ element of the Hessian of $\log F$ at $\mathbf{x}_{\max}$, are

$$\begin{aligned}
h_{11} &= -(160\rho^4 d - 256\rho^4 d^2 - 32\rho^4 + 128d^3\rho^4 + 352\rho^3 d^2 - 192d^3\rho^3 - 192d\rho^3 \\
&\quad +32\rho^3 + 32\rho^2 d + 76d^3\rho^2 - 104\rho^2 d^2 - 12d^3\rho + 8d^2\rho + d^3) \\
&\quad /(2\rho^2(-d-2\rho+4d\rho)^2(-1+2\rho)^2), \\
h_{22} &= -(32d^4\rho^4 - 48d^4\rho^3 + 12d^4\rho^2 - 128d^3\rho^4 + 176d^3\rho^3 - 36d^3\rho^2 - 4d^3\rho + d^3 \\
&\quad +208\rho^4 d^2 - 272\rho^3 d^2 + 68\rho^2 d^2 - 4d^2\rho - 112\rho^4 d + 128d\rho^3 - 20\rho^2 d + 16\rho^4 \\
&\quad -16\rho^3)/(4(d-1)\rho^2(-d-2\rho+4d\rho)^2(-1+2\rho)^2), \\
h_{33} &= -(d(8\rho^2 d^2 - 4d^2\rho + d^2 - 8\rho^2 d + 4\rho^2))/(4\rho^2(d-1)^2(-d-2\rho+4d\rho)^2), \\
h_{12} &= -2(8d^3\rho^3 + 20d\rho^3 - 28\rho^3 d^2 - 4\rho^3 - 12d^3\rho^2 + 40\rho^2 d^2 - 24\rho^2 d + 4\rho^2 \\
&\quad +3d^3\rho - 12d^2\rho + 4d\rho + d^2)/(\rho(-d-2\rho+4d\rho)^2(-1+2\rho)^2), \\
h_{13} &= 2(d(-d+2d\rho-\rho))/((-d-2\rho+4d\rho)^2\rho), \\
h_{23} &= ((-d+2d\rho-2\rho)d)/(\rho(-d-2\rho+4d\rho)^2).
\end{aligned}$$

Thus we obtain

$$\sum_R \alpha(\mathbf{x})F(\mathbf{x})^n = \alpha(\mathbf{x}_{\max})F(\mathbf{x}_{\max})^n \sum_B \exp\left(-n\left(as_1^2+bs_2^2+cs_3^2+es_1s_2+fs_1s_3+gs_2s_3\right)\right),$$

with $\alpha(\mathbf{x}_{\max}) = \frac{d^4(1-2\rho)\sqrt{d(d-1)}}{4(d-1)^2\rho^4(d-(4d-2)\rho)^3}$. Since $n \to \infty$, the range of integration may be extended to $\pm\infty$. Hence the sum is asymptotic to

$$n^{3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\left(at_1^2+bt_2^2+ct_3^2+et_1t_2+ft_1t_3+gt_2t_3\right)\right)dt_1dt_2dt_3,$$

where

$$t_1 = \frac{(x-2\rho^2/d)n}{\sqrt{n}} \quad t_2 = \frac{(y-4(d-1)\rho^2/d)n}{\sqrt{n}} \quad t_3 = \frac{(z-4(d-1)^2\rho^2/d)n}{\sqrt{n}}.$$

The evaluation of the triple integral yields

$$\frac{2^4\pi^{3/2}(d-1)\rho^3(1-2\rho)(d-(4d-2)\rho)}{d}\sqrt{\frac{d-1}{d((-4d^3+12d^2-12d+8)\rho^2-4d\rho+d^2)}}.$$

Thus, multiplying these transforms (8) into

$$\begin{aligned}\mathbf{E}Y_{d,\rho}(Y_{d,\rho}-1) \sim{}& \frac{d^3(1-2\rho)^2}{2\pi n\rho(d-(4d-2)\rho)^2}\\ &\times\sqrt{\frac{1}{(-4d^3+12d^2-12d+8)\rho^2-4d\rho+d^2}}\\ &\times\left(\frac{(1-2\rho)^{2(d-1)(1-2\rho)}d^{d-(4d-4)\rho}}{2^{2\rho}\rho^{2\rho}(d-(4d-2)\rho)^{d-(4d-2)\rho}}\right)^n. \end{aligned} \tag{15}$$

Note that $\mathbf{E}Y_{d,\rho} \to \infty$ implies $\mathbf{E}Y_{d,\rho}(Y_{d,\rho}-1) \sim \mathbf{E}Y_{d,\rho}^2$ and $\mathbf{Var}Y_{d,\rho} = \mathbf{E}Y_{d,\rho}^2 - (\mathbf{E}Y_{d,\rho})^2$. Thus Equation (15) gives the desired result and this completes the proof of the theorem. ∎

3 Expectation Conditioned on Cycle Distribution

Lemma 1. *Let $d \geq 3$ and $0 < \tau < d/(4d-2)$ be fixed. Let $X_k = X_k(n)$ denote the number of cycles of length k in $P \in \mathcal{P}_{n,d}$. Then for any finite sequence $j_1, \ldots, j_l$ of non-negative integers,*

$$\frac{\mathbf{E}\left(Y_{d,\rho}[X_1]_{j_1}\cdots[X_l]_{j_l}\right)}{\mathbf{E}Y_{d,\rho}} \to \prod_{k=1}^{l}\left(\lambda_k\left(1+\delta_k\right)\right)^{j_k} \quad \text{as } n \to \infty;$$

with $\lambda_k = \frac{(d-1)^k}{2k}$ and $\delta_k = 2\,\mathrm{Re}(\mu^k)$ where $\mu = \frac{-(d-1)\rho+\sqrt{(d^2+2d+1)\rho^2-2d\rho}}{d(1-2\rho)}$.

Proof. We will first prove

$$\frac{\mathbf{E}\left(Y_{d,\rho}X_k\right)}{\mathbf{E}Y_{d,\rho}} \sim \lambda_k\left(1+\delta_k\right). \tag{16}$$

The number of ways to choose an oriented cycle of length k in the pairing, with a root vertex, is

$$\frac{n!(d(d-1))^k}{(n-k)!}. \tag{17}$$

Let C denote the set of pairs corresponding to an oriented and rooted k-cycle, and define $\mathcal{M}$ to be the set of pairs corresponding to an induced matching of size $p = \rho n$. Given a fixed C, suppose $C \cap \mathcal{M}$ consists of s_1 independent edges and s_2 vertices (these vertices are the end-points of edges in $\mathcal{M}$). The vertices of C can be classified into 3 types. The first are vertices which are not lying on $\mathcal{M}$. We denote this type of vertices by 0. The remaining vertices in C are the end-points of edges in $\mathcal{M}$. They are either preceded by an edge in $\mathcal{M}$ or preceded by an edge not in $\mathcal{M}$ and we denote them by 1 and 2 respectively. If we walk along C from the root vertex then we obtain a sequence $S_0 \in \{0,1,2\}^k$.

Given fixed C and S_0, we observe that in order to create the remaining $p - s_1$ independent edges, $2(p-s_1) - s_2$ vertices need to be chosen from the remaining $n-k$ vertices. Hence, the number of ways to complete the $p - s_1$ independent edges, including with the points used, is

$$\begin{aligned}&\binom{n-k}{2p-2s_1-s_2}\frac{(2p-2s_1)!}{2^{p-s_1}(p-s_1)!}d^{2p-2s_1-s_2}(d-2)^{s_2}\\ &= \frac{(n-k)!(2p-2s_1)!d^{2p-2s_1-s_2}(d-2)^{s_2}}{2^{p-s_1}(2p-2s_1-s_2)!(n-k-2p+2s_1+s_2)!(p-s_1)!}.\end{aligned} \tag{18}$$

Having completed $\mathcal{M}$, we note that the number of ways to choose the points in the pairing corresponding to the edges from $\mathcal{M}$ to the other $n-2p$ vertices is

$$\begin{aligned}&[dn-2dp-2k+4s_1+2s_2]_{2(d-1)p-2s_1-2s_2}\\ &= \frac{(dn-2dp-2k+4s_1+2s_2)!}{(dn-(4d-2)p-2k+6s_1+4s_2)!}.\end{aligned} \tag{19}$$

The number of ways to complete the pairing given fixed C and S_0 is

$$N(dn-(4d-2)p-2k+6s_1+4s_2). \tag{20}$$

Multiplying equations (18)–(20), summing over all possible S_0 and then multiplying by (17) we have

$$\begin{aligned}\sum_{S_0}\Bigg[&\frac{n!(2p-2s_1)!d^{2p-2s_1-s_2}(d-2)^{s_2}(d(d-1))^k}{2^{p-s_1}(p-s_1)!(2p-2s_1-s_2)!(n-k-2p+2s_1+s_2)!}\\ &\times\frac{(dn-2dp-2k+4s_1+2s_2)!}{(dn-(4d-2)p-2k+6s_1+4s_2)!}N(dn-(4d-2)p-2k+6s_1+4s_2)\Bigg],\end{aligned}$$

which is the number of pairings containing an induced matching of size p and an oriented and rooted cycle of length k. Dividing this by the number of pairings containing an induced matching of size p as in (2), and then evaluating this asymptotically we have

$$\left(\frac{(d-1)(d-(4d-2)\rho)}{d(1-2\rho)}\right)^k$$
$$\times \sum_{S_0} \left(\frac{2d^2\rho(1-2\rho)^2}{(d-(4d-2)\rho)^3}\right)^{s_1} \left(\frac{2d(d-2)\rho(1-2\rho)}{(d-(4d-2)\rho)^2}\right)^{s_2}. \qquad (21)$$

We consider 0, 1 and 2 as three states in a process similar to a Markov Chain where the final state is equal to the initial state. We observe here that in S_0

(i) 1 must be followed by 0 and this contributes a factor $\frac{2d^2\rho(1-2\rho)^2}{(d-(4d-2)\rho)^3}$,
(ii) 2 followed by 0 contributes a factor $\frac{2d(d-2)\rho(1-2\rho)}{(d-(4d-2)\rho)^2}$.

The transition matrix is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ \frac{2d^2\rho(1-2\rho)^2}{(d-(4d-2)\rho)^3} & 0 & 0 \\ \frac{2d(d-2)\rho(1-2\rho)}{(d-(4d-2)\rho)^2} & 1 & 0 \end{pmatrix}.$$

Thus, we can rewrite

$$\sum_{S_0} \left(\frac{(d-1)(1-(d+1)\rho)}{(d-2)^2(1-\rho)}\right)^{s_1} \left(\frac{2d(d-2)\rho(1-2\rho)}{(d-(4d-2)\rho)^2}\right)^{s_2} = Tr\left(\mathbf{A}^k\right).$$

Noting that $\bar{\mu}$ is the conjugate of μ, we obtain that the eigenvalues of $\mathbf{A}$ are $\gamma_1 = \frac{d(1-2\rho)}{d-(4d-2)\rho}$, $\gamma_2 = \gamma_1\mu$ and $\gamma_3 = \gamma_1\bar{\mu}$. Hence,

$$Tr\left(\mathbf{A}^k\right) = (\gamma_1)^k + (\gamma_2)^k + (\gamma_3)^k.$$

This and noting that $\delta_k = \mu^k + \bar{\mu}^k$ transform (21) into

$$(d-1)^k (1+\delta_k).$$

Finally, dividing by $2k$ to remove the orientation and rooting of the cycle we obtain (16). The argument also works for higher moments. The proof is complete. ∎

Remark 1. From Lemma 1 we find that

$$\delta_k = 2\mathrm{Re}(\mu^k) = 2\left(\frac{2\rho}{d(1-2\rho)}\right)^{k/2} \cos(k\theta),$$

where $\theta = -\arctan(\sqrt{2d\rho - (d^2+2d+1)\rho^2}/((d-1)\rho)) + \pi$.

It will be shown that if $d \geq 3$ then $\delta_k > -1$ for $\rho \in (0, d/(4d-2))$ and all $k \geq 1$. Assume first that $k \geq 2$. We note that $(2\rho)/(d(1-2\rho)) < 1/2$, for $d \geq 3$ and for ρ in the open interval. Since this factor is raised to a power at least 1, the claim follows immediately. For $k = 1$, recalling that $\bar{\mu}$ is the conjugate of μ, we write $\delta_1 = \mu + \bar{\mu}$. It is easy to see that $\delta_1 = -2(d-1)\rho/(d-2d\rho) > -1$ for all $d \geq 3$ and for $\rho \in (0, d/(4d-2))$. This completes the claim.

4 Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1: First we will show the corresponding result in $\mathcal{P}_{n,d}$.

Theorem 4. *Let $3 \le d \le 5$ be fixed. If $\tau = \beta_1(d)$ then for $P \in \mathcal{P}_{n,d}$, $G(P)$ a.a.s. has an induced matching of size ρn. Moreover,*

$$\frac{Y_{d,\rho}}{\mathbf{E}Y_{d,\rho}} \xrightarrow{d} W = \Pi_{k=1}^{\infty} (1+\delta_k)^{Z_k} e^{-\lambda_k \delta_k} \ \textit{as}\ n \to \infty,$$

where Z_k are independent Poisson variables with $\mathbf{E}Z_k = \lambda_k$ for $k \ge 1$, where λ_k and δ_k as in Lemma 1.

Proof. We only have to show that the conditions (i)–(iv) in [16–Theorem 4.1] are fulfilled by the variable $Y_{d,\rho}$. Note that $\sum_k \lambda_k \delta_k^2 = \log C(d,\rho)$ with $C(d,\rho)$ as in Theorem 3. Hence Theorem 2, Theorem 3 and Lemma 1 (see Remark 1) complete the proof. ∎

Since the claim in Theorem 4 is a.a.s true conditioned on no loops or multiples edges (see Bollobás [4]) Theorem 1 follows immediately. Moreover, from the argument in [12–page 375] we also obtain

$$\frac{\mathbf{E}Y'_{d,\rho}}{\mathbf{E}Y_{d,\rho}} \sim \exp\left(\frac{(d-1)^2\rho}{d(1-2\rho)} - \frac{(d-1)^2\rho(d^2\rho+\rho-d)}{d^2(1-2\rho)^2}\right) \text{ and}$$

$$\frac{\mathbf{E}{Y'_{d,\rho}}^2}{(\mathbf{E}Y_{d,\rho})^2} \sim \exp\left(-\frac{2(d-1)^3\rho^2}{d^2(1-2\rho)^2} - \frac{4(d-1)^2\rho^2(d^2\rho+\rho-d)^2}{d^4(1-2\rho)^4}\right)\frac{\mathbf{E}{Y_{d,\rho}}^2}{(\mathbf{E}Y_{d,\rho})^2}. \qquad ∎$$

Proof of Corollary 1: Since $\mathbf{E}Y_{d,\rho}$ is proportional to $\mathbf{E}Y'_{d,\rho}$, the a.a. sure upper bound is straightforward from the last part of Theorem 2. The a.a. sure lower bound comes immediately from Theorem 1.

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Antimagic Valuations for the Special Class of Plane Graphs

Martin Bača[1], Edy Tri Baskoro[2], and Mirka Miller[3]

[1] Department of Appl. Mathematics,
Technical University, Letná 9, 042 00 Košice, Slovak Republic
Martin.Baca@tuke.sk

[2] Department of Mathematics, Institut Teknologi Bandung, Indonesia
ebaskoro@dns.math.itb.ac.id

[3] School of Information Technology and Mathematical Sciences,
University of Ballarat, VIC 3350, Australia
m.miller@ballarat.edu.au

Abstract. We deal with the problem of labeling the vertices, edges and faces of a special class of plane graphs with 3-sided internal faces in such a way that the label of a face and the labels of the vertices and edges surrounding that face all together add up to the weight of that face. These face weights then form an arithmetic progression with common difference d.

1 Introduction

All graphs in this paper will be finite, connected and plane. The graph $G = G(V, E, F)$ has vertex set $V = V(G)$, edge set $E = E(G)$ and face set $F = F(G)$. The cardinality of a set A will be denoted by $|A|$. For a general reference of graph-theoretic notions, see [9] and [10].

A labeling (or valuation) of a graph is any map that carries some set of graph elements to numbers (usually to the positive or non-negative integers).

A labeling of *type* $(1, 1, 1)$ is a one-to-one map from $V \cup E \cup F$ onto the integers $\{1, 2, 3, ..., |V(G)| + |E(G)| + |F(G)|\}$.

If we label only vertices or only edges or only faces, we call such a labeling a *vertex labeling*, an *edge labeling* or a *face labeling*, respectively.

The *weight* $w(x)$ of a face x under a labeling is the sum of the labels (if present) carried by that face and the edges and vertices surrounding it.

A labeling of plane graph G is called *d-antimagic* if for every number s the set of s-sided face weights is $W_s=\{a_s, a_s + d, a_s + 2d, ..., a_s + (f_s - 1)d\}$ for some integers a_s and d, $d \geq 0$, where f_s is the number of s-sided faces. We allow different sets W_s for different s.

The d-antimagic labeling of type $(1, 1, 1)$ was introduced in [5] and it is natural extension of the notions of *magic* (0-antimagic) labeling and *consecutive* (1-antimagic) labeling defined by Ko Wei Lih in [7].

Ko Wei Lih [7] described magic (0-antimagic) and consecutive (1-antimagic) labelings for *wheels*, *friendship graphs* and *prisms*.

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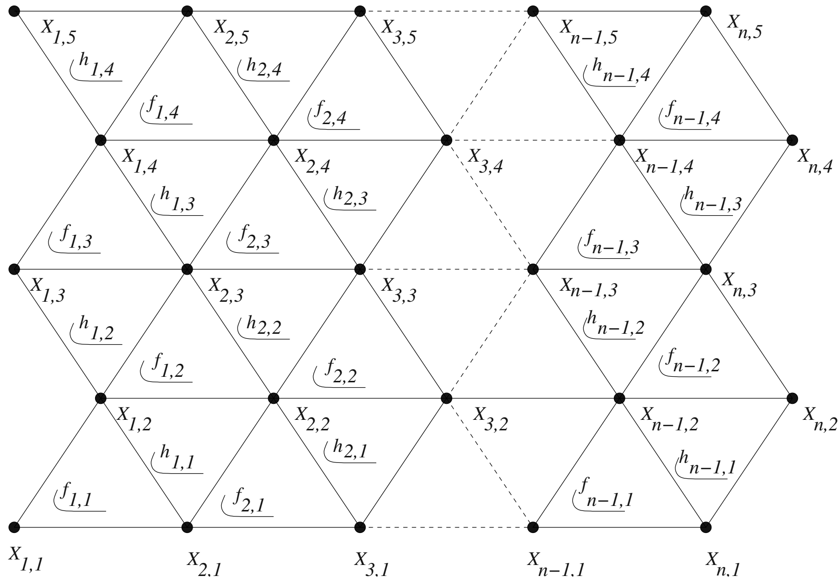

Fig. 1. The labeled graph L_n^m for $m = 4$

Magic (0-antimagic) labelings of type $(1, 1, 1)$ for *grid graphs* and *honeycombs* are given in [3] and [4].

Many other researchers investigated different forms of antimagic graphs. For example, see Hartsfield and Ringel [6], Wagner and Bodendiek [8].

Let $I = \{1, 2, ..., n\}$ and $J = \{1, 2, ..., m + 1\}$ be index sets.

For $n \geq 2$, $1 \leq m \leq 4$, let L_n^m be the graph with the vertex set $V(L_n^m) = \{x_{i,j} : i \in I \text{ and } j \in J\}$ and the edge set

$E(L_n^m) = \{x_{i,j}x_{i+1,j} : i \in I - \{n\} \text{ and } j \in J\}$
$\cup\{x_{i,j}x_{i,j+1} : i \in I \text{ and} j \in J - \{m + 1\}\}$
$\cup\{x_{i+1,j}x_{i,j+1} : i \in I - \{n\}, j \in J - \{m + 1\} \text{ and } j \text{ is odd }\}$
$\cup\{x_{i,j}x_{i+1,j+1} : i \in I - \{n\}, j \in J - \{m + 1\} \text{ and } j \text{ is even }\}$,

embeded in the plane and labeled as in Fig. 1 (if $m = 4$).

The face set $F(L_n^m)$ contains $|F(L_n^m)| - 1 = 2(n - 1)m$ 3-sided faces and one external infinite face. $|V(L_n^m)| = n(m + 1)$, $|E(L_n^m)| = |V(L_n^m)| + |F(L_n^m)| - 2$.

Magic (0-antimagic) labelings of type $(1, 1, 1)$ of plane graphs L_n^m for $n \geq 2$, $m = 1$ are described in [1] and for $n \geq 2$, $2 \leq m \leq 3$ are given in [2].

In this paper are studied the properties of d-antimagic labelings of L_n^m and it is shown how to construct d-antimagic labelings of L_n^m for $n \geq 2$, $1 \leq m \leq 4$, $d \in \{0, 2, 4\}$.

2 Elementary Counting

In this section we shall find bounds for a feasible value d for the vertex labeling and the edge labeling of L_n^m.

Assume that there exists a bijection from $V(L_n^m)$ onto the integers $\{1, 2, 3, ..., |V(L_n^m)|\}$ which is d-antimagic and $W_3 = \{w(x) : x \in F(L_n^m)\} = \{a_3, a_3 + d, ..., a_3 + (f_3 - 1)d\}$ is the set of 3-sided face weights. The minimum possible weight of a 3-sided face in d-antimagic vertex labeling is at least 6. On the other hand, the maximum weight of a 3-sided face is no more than $\sum_{i=1}^{3}(|V(L_n^m)|+1-i)$.

Thus we have

$$a_3 + (f_3 - 1)d \leq \sum_{i=1}^{3}(|V(L_n^m)| + 1 - i)$$

and

$$d \leq \frac{3nm + 3n - 3 - a_3}{2nm - 2m - 1} \leq \frac{3nm + 3n - 9}{2nm - 2m - 1} \leq 3 \ .$$

This implies

Lemma 1. *For every plane graph L_n^m, $n \geq 2$, $m \geq 1$, there is no d-antimagic vertex labeling with $d > 3$.*

For d-antimagic edge labeling, the maximum 3-sided face weight is no more than $\sum_{i=1}^{3}(|E(L_n^m)| + 1 - i)$.

Hence

$$a_3 + (f_3 - 1)d \leq \sum_{i=1}^{3}(|E(L_n^m)| + 1 - i) \ .$$

If $a_3 \geq 6$, $n \geq 2$, $m \geq 1$, we get

$$d \leq \frac{9nm + 3n - 6m - 12}{2nm - 2m - 1} \leq 6 \ .$$

It follows that

Lemma 2. *For every plane graph L_n^m, $n \geq 2$, $m \geq 1$, there is no d-antimagic edge labeling whenever $d > 6$.*

Applying Lemma 1, Lemma 2 and the fact that under d-antimagic face labeling $F(L_n^m) \to \{1, 2, ..., |F(L_n^m)|\}$ the parameter d is no more than 1, we obtain

Theorem 1. *Let L_n^m, $n \geq 2$, $m \geq 1$, be a plane graph which admits d_1-antimagic vertex labeling g_1, d_2-antimagic edge labeling g_2 and 1-antimagic face labeling g_3, $d_1 \geq 0$, $d_2 \geq 0$. If the labelings g_1, $|V(L_n^m)| + g_2$ and $|V(L_n^m)| + |E(L_n^m)| + g_3$ combine into a d-antimagic labeling of type $(1, 1, 1)$ then the parameter $d \leq 10$.*

3 Labelings

Consider the plane graph L_n^m, $n \geq 2$, $1 \leq m \leq 4$, and construct the vertex labeling g_1 and the edge labeling g_2 in the following way.

$$g_1(x_{i,j}) = n(m+2-j)+1-i \qquad \text{if } i \in I \text{ and } j \in J,$$

$$g_2(x_{i,j}x_{i+1,j}) = (n-1)j+1-i \qquad \text{if } i \in I-\{n\} \text{ and } j \in J,$$

$$g_2(x_{i,j}x_{i,j+1}) =$$
$$= \begin{cases} m(n-1)+n(j+1)-2+i & \text{if } i \in I \text{ and } j \leq 2 \\ m(n-1)+n(j+3)-3+i & \text{if } i \in I \text{ and } j \in J-\{m+1\}, j \geq 3, \end{cases}$$

$$g_2(x_{i+1,j}x_{i,j+1}) = (n-1)(m+1)+\tfrac{j-1}{2}(4n-1)+i$$
$$\text{if } i \in I-\{n\},\ j \in J-\{m+1\} \text{ and } j \text{ is odd},$$

$$g_2(x_{i,j}x_{i+1,j+1}) = m(n-1)+\tfrac{j}{2}(4n-1)-1+i$$
$$\text{if } i \in I-\{n\},\ j \in J-\{m+1\} \text{ and } j \text{ is even}.$$

Let us denote the weights of the 3-sided faces of L_n^m under a vertex labeling g_1 and an edge labeling g_2 as follows:

If j is odd, $j \in J-\{m+1\}$ and $i \in I-\{n\}$ then

$w_{g_1}(f_{i,j}) = g_1(x_{i,j}) + g_1(x_{i+1,j}) + g_1(x_{i,j+1})$,

$w_{g_1}(h_{i,j}) = g_1(x_{i+1,j}) + g_1(x_{i,j+1}) + g_1(x_{i+1,j+1})$,

$w_{g_2}(f_{i,j}) = g_2(x_{i,j}x_{i+1,j}) + g_2(x_{i,j}x_{i,j+1}) + g_2(x_{i+1,j}x_{i,j+1})$ and

$w_{g_2}(h_{i,j}) = g_2(x_{i+1,j}x_{i,j+1}) + g_2(x_{i,j+1}x_{i+1,j+1}) + g_2(x_{i+1,j}x_{i+1,j+1})$.

If j is even, $j \in J-\{m+1\}$ and $i \in I-\{n\}$ then

$v_{g_1}(f_{i,j}) = g_1(x_{i,j}) + g_1(x_{i+1,j}) + g_1(x_{i+1,j+1})$,

$v_{g_1}(h_{i,j}) = g_1(x_{i,j}) + g_1(x_{i,j+1}) + g_1(x_{i+1,j+1})$,

$v_{g_2}(f_{i,j}) = g_2(x_{i,j}x_{i+1,j}) + g_2(x_{i,j}x_{i+1,j+1}) + g_2(x_{i+1,j}x_{i+1,j+1})$ and

$v_{g_2}(h_{i,j}) = g_2(x_{i,j}x_{i,j+1}) + g_2(x_{i,j+1}x_{i+1,j+1}) + g_2(x_{i,j}x_{i+1,j+1})$.

4 The Main Results

Theorem 2. *If $n \geq 2$, $1 \leq m \leq 4$ and $d \in \{0,2\}$, then the plane graph L_n^m has a d-antimagic labeling of type $(1,1,1)$.*

Proof. It is not difficult to check that the values of g_1 are $1, 2, ..., |V(L_n^m)|$.

If $i \in I$, $j \in J$ and $m \leq 2$, then the edge labeling g_2 uses each integer $1, 2, ..., |E(L_n^m)|$ exactly once.

If $i \in I$, $j \in J$ and $m \geq 3$, then the edge labeling g_2 successively attain the values $1, 2, ..., m(n-1)+5n-4, m(n-1)+5n-3$ and the values $m(n-1)+5n-1, m(n-1)+5n, ..., |E(L_n^m)|+1$.

Label the vertices and the edges of L_n^m, $n \geq 2$, $1 \leq m \leq 4$, by the vertex labeling g_1 and the edge labeling g_2. By direct computation we obtain that *a)* if j is odd, $j \in J-\{m+1\}$ and $i \in I-\{n\}$ then the weights of the 3-sided faces (under the vertex labeling g_1 and the edge labeling g_2) are

$$w_{g_1}(f_{i,j}) = n(3m+5-3j)+2-3i,$$

$$w_{g_1}(h_{i,j}) = n(3m+4-3j)+1-3i,$$

$$w_{g_2}(f_{i,j}) = n(2m-1)-2m+(5n-2)j-1+i,$$

$$w_{g_2}(h_{i,j}) = 2m(n-1)+(5n-2)j-1+i,$$

b) if j is even, $j \in J-\{m+1\}$ and $i \in I-\{n\}$ then the weights of the 3-sided faces are

$$v_{g_1}(f_{i,j}) = n(3m+5-3j)+1-3i,$$

$$v_{g_1}(h_{i,j}) = n(3m+4-3j)+2-3i,$$

$$v_{g_2}(f_{i,j}) = n(2m-1)-2m+(5n-2)j+i,$$

$$v_{g_2}(h_{i,j}) = 2m(n-1)+(5n-2)j-2+i.$$

If we label the vertices and the edges of L_n^m by g_1 and $|V(L_n^m)|+g_2$, respectively, then for $n \geq 2$ and $m \leq 2$ we obtain a resulting labeling with values $1,2,...,|V(L_n^m)|, |V(L_n^m)|+1, |V(L_n^m)|+2,...,|V(L_n^m)|+|E(L_n^m)|$ and for $n \geq 2$ and $3 \leq m \leq 4$ we have a resulting labeling with values $1,2,...,|V(L_n^m)|, |V(L_n^m)|$ $+1, |V(L_n^m)|+2,...,|V(L_n^m)|+m(n-1)+5n-4, |V(L_n^m)|+m(n-1)+5n$ $-3, |V(L_n^m)|+m(n-1)+5n-1, |V(L_n^m)|+m(n-1)+5n,...,|V(L_n^m)|+|E(L_n^m)|+1$.

We can see that the weights of all 3-sided faces under the resulting labeling constitute a set W of consecutive integers

$$\begin{aligned} W = &\{w_{g_1}(f_{i,j})+3|V(L_n^m)|+w_{g_2}(f_{i,j}) : i \in I-\{n\}, j \in J-\{m+1\} \text{ and } j \text{ is odd}\} \\ &\cup\{w_{g_1}(h_{i,j})+3|V(L_n^m)|+w_{g_2}(h_{i,j}) : i \in I-\{n\}, j \in J-\{m+1\} \text{ and } j \text{ is odd}\} \\ &\cup\{v_{g_1}(f_{i,j})+3|V(L_n^m)|+v_{g_2}(f_{i,j}) : i \in I-\{n\}, j \in J-\{m+1\} \text{ and } j \text{ is even}\} \\ &\cup\{v_{g_1}(h_{i,j})+3|V(L_n^m)|+v_{g_2}(h_{i,j}) : i \in I-\{n\}, j \in J-\{m+1\} \text{ and } j \text{ is even}\} \\ &= \{5|V(L_n^m)|+|E(L_n^m)|-n+2j(n-1)+2+i : i \in I-\{n\}, j \in J-\{m+1\}\} \\ &\cup\{5|V(L_n^m)|+|E(L_n^m)|+2j(n-1)+1+i : i \in I-\{n\}, j \in J-\{m+1\}\}. \end{aligned}$$

Complete the face labeling g_3 of L_n^m so that the values $|V(L_n^m)|+|E(L_n^m)|$ $+2, |V(L_n^m)|+|E(L_n^m)|+3,...,|V(L_n^m)|+|E(L_n^m)|+|F(L_n^m)|$ will be given to the 3-sided faces and the external infinite face receives the value $|V(L_n^m)|+|E(L_n^m)|+1$ (if $n \geq 2$, $m \leq 2$) or the value $|V(L_n^m)|+m(n-1)+5n-2$ (if $n \geq 2$, $3 \leq m \leq 4$).

We are able to arrange the face values of the 3-sided faces so that the labelings g_1, $|V(L_n^m)|+g_2$ and g_3 combined together give a labeling of type $(1,1,1)$, where

(i) all 3-sided faces have the same weight $6|V(L_n^m)|+2|E(L_n^m)|+|F(L_n^m)|+$ $n+1$ or
(ii) the weights of the 3-sided faces form an arithmetic progression of difference $d=2$. □

Theorem 3. *If* $n \geq 2$, $1 \leq m \leq 4$, *then the plane graph* L_n^m *has a* 4*-antimagic labeling of type* $(1,1,1)$.

Proof. Define new labelings g_4 and g_5 such that

$$g_4(x) = 2g_1(x) - 1 \qquad \text{for every vertex } x \in V(L_n^m) \text{ and}$$

$$g_5(xy) = 2g_2(xy) \qquad \text{for every edge } xy \in E(L_n^m).$$

The labeling g_4 uses the values in the set $\{1, 3, 5, ..., 2|V(L_n^m)| - 1\}$.

If $i \in I$, $j \in J$ and $m \leq 2$, then the labeling g_5 uses the values $2, 4, 6, ..., 2|E(L_n^m)|$.

If $i \in I$, $j \in J$ and $m \geq 3$, then the labeling g_5 receives the values $2, 4, 6, ..., 2m(n-1) + 10n - 8, 2m(n-1) + 10n - 6, 2m(n-1) + 10n - 2, 2m(n-1) + 10n, ..., 2|E(L_n^m)| + 2$.

Label the vertices and the edges of L_n^m, $n \geq 2$, $1 \leq m \leq 4$, by the labeling g_4 and g_5, respectively.

a) If j is odd, $j \in J - \{m+1\}$ and $i \in I - \{n\}$ then

$$w_{g_4}(f_{i,j}) = 2n(3m + 5 - 3j) + 1 - 6i,$$

$$w_{g_4}(h_{i,j}) = 2n(3m + 4 - 3j) - 1 - 6i,$$

$$w_{g_5}(f_{i,j}) = 2n(2m - 1) - 4m + 2(5n - 2)j - 2 + 2i,$$

$$w_{g_5}(h_{i,j}) = 4m(n - 1) + 2(5n - 2)j - 2 + 2i.$$

b) If j is even, $j \in J - \{m+1\}$ and $i \in I - \{n\}$ then

$$v_{g_4}(f_{i,j}) = 2n(3m + 5 - 3j) - 1 - 6i,$$

$$v_{g_4}(h_{i,j}) = 2n(3m + 4 - 3j) + 1 - 6i,$$

$$v_{g_5}(f_{i,j}) = 2n(2m - 1) - 4m + 2(5n - 2)j + 2i,$$

$$v_{g_5}(h_{i,j}) = 4m(n - 1) + 2(5n - 2)j - 4 + 2i.$$

The labelings g_4 and g_5 combine to give a labeling where the weights of all 3-sided faces constitute the set

$$\begin{aligned} V = &\{w_{g_4}(f_{i,j}) + w_{g_5}(f_{i,j}) : i \in I - \{n\}, j \in J - \{m+1\} \text{ and } j \text{ is odd }\} \\ &\cup \{w_{g_4}(h_{i,j}) + w_{g_5}(h_{i,j}) : i \in I - \{n\}, j \in J - \{m+1\} \text{ and } j \text{ is odd }\} \\ &\cup \{v_{g_4}(f_{i,j}) + v_{g_5}(f_{i,j}) : i \in I - \{n\}, j \in J - \{m+1\} \text{ and } j \text{ is even }\} \\ &\cup \{v_{g_4}(h_{i,j}) + v_{g_5}(h_{i,j}) : i \in I - \{n\}, j \in J - \{m+1\} \text{ and } j \text{ is even }\} \\ &= \{4|V(L_n^m)| + 2|E(L_n^m)| - 2n + 1 + 4j(n-1) + 2i : i \in I - \{n\}, j \in J - \{m+1\}\} \\ &\cup \{4|V(L_n^m)| + 2|E(L_n^m)| - 1 + 4j(n-1) + 2i : i \in I - \{n\}, j \in J - \{m+1\}\}. \end{aligned}$$

The elements of V form an arithmetic progression of difference 2.

Now, if $m \leq 2$, consider the labeling of faces of L_n^m with values in the set $\{2|V(L_n^m)| + 1, 2|V(L_n^m)| + 3, ..., |V(L_n^m)| + |E(L_n^m)| + |F(L_n^m)|\}$ where the external infinite face receives the value $|V(L_n^m)| + |E(L_n^m)| + |F(L_n^m)|$. Arrange the other values to the 3-sided faces in such a way that combining the face labeling and labelings g_4 and g_5 yields a labeling of type $(1,1,1)$ which is 4-antimagic.

Similarly, if $3 \leq m \leq 4$, complete the face labeling of L_n^m in such a way that the values $2|V(L_n^m)|+1, 2|V(L_n^m)|+3, ..., |V(L_n^m)| + |E(L_n^m)| + |F(L_n^m)| - 1$ will be given to the 3-sided faces and value $2m(n-1) + 10n - 4$ to the external infinite face. We can arrange the face labeling of the 3-sided faces so that the face labeling and the labelings g_4 and g_5 give together a 4-antimagic labeling of type $(1,1,1)$. □

5 Conclusion

In the foregoing section we studied d-antimagic labelings for plane graphs L_n^m and proved that for $n \geq 2$, $1 \leq m \leq 4$, $d \in \{0,2,4\}$ there exist d-antimagic labelings of type $(1,1,1)$.

We have shown a bound for the feasible values of the parameter d. We conclude with the following open problem.

Open Problem. *Find other possible values of the parameter d and corresponding d-antimagic labelings of type $(1,1,1)$ for L_n^m, $n \geq 2$, $1 \leq m \leq 4$ (or $m \geq 1$).*

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A General Framework for Coloring Problems: Old Results, New Results, and Open Problems

Hajo Broersma*

Department of Applied Mathematics, Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands
broersma@math.utwente.nl

Abstract. In this survey paper we present a general framework for coloring problems that was introduced in a joint paper which the author presented at WG2003. We show how a number of different types of coloring problems, most of which have been motivated from frequency assignment, fit into this framework. We give a survey of the existing results, mainly based on and strongly biased by joint work of the author with several different groups of coauthors, include some new results, and discuss several open problems for each of the variants.

Keywords: graph coloring; graph partitioning; forbidden subgraph; planar graph; computational complexity.

AMS Subject Classifications: 05C15,05C85,05C17

1 General Introduction

In the application area of frequency assignment graphs are used to model the topology and mutual interference between transmitters (receivers, base stations): the vertices of the graph represent the transmitters; two vertices are adjacent in the graph if the corresponding transmitters are so close (or so strong) that they are likely to interfere if they broadcast on the same or 'similar' frequency channels. The problem in practice is to assign a limited number of frequency channels in an economical way to the transmitters in such a way that interference is kept at an 'acceptable level'. This has led to various different types of coloring problems in graphs, depending on different ways to model the level of interference, the notion of similar frequency channels, and the definition of acceptable level of interference (See e.g. [25],[32]).

In [10] an attempt was made to capture a number of different coloring problems in a unifying model. This general framework that we will consider here too is as follows:

* Present address: Department of Theoretical Computer Science, University of Durham, Science Laboratories, South Road, Durham DH1 3LE, England.

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> Given two graphs G_1 and G_2 with the property that G_1 is a (spanning) subgraph of G_2, one considers the following type of coloring problems: Determine a coloring of (G_1 and) G_2 that satisfies certain restrictions of type 1 in G_1, and restrictions of type 2 in G_2, using a limited number of colors.

Many known coloring problems related to frequency assignment fit into this general framework. We will discuss the following types of problems.

1.1 Distant-2 Coloring

First of all suppose that $G_2 = G_1^2$, i.e. G_2 is obtained from G_1 by adding edges between all pairs of vertices that are at distance 2 in G_1. If one just asks for a proper vertex coloring of G_2 (and G_1), this is known as the distant-2 coloring problem. Much of the research has been concentrated on the case that G_1 is a planar graph, and on obtaining good upper bounds in terms of the maximum degree of G_1 for the minimum number of colors needed in this case. In Section 2 we will survey some of the existing results, and discuss the proof techniques and open problems in this subarea.

1.2 Radio Coloring

In some versions of the previous problem one puts the additional restriction on G_1 that the colors should be sufficiently separated. One way to model this is to use positive integers for the colors (modeling certain frequency channels) and to ask for a coloring of G_1 and G_2 such that the colors on adjacent vertices in G_2 are different, whereas they differ by at least 2 on adjacent vertices in G_1. This problem is known as the radio coloring problem and has been studied under various names. In Section 3 we will briefly survey some of the existing results in this subarea.

1.3 Radio Labeling

The so-called radio labeling problem models a practical setting in which all assigned frequency channels should be distinct, with the additional restriction that adjacent transmitters should use sufficiently separated frequency channels. Within the above framework this can be modeled by considering the graph G_1 that models the adjacencies of n transmitters, and taking $G_2 = K_n$, the complete graph on n vertices. The restrictions are clear: one asks for a proper vertex coloring of G_2 such that adjacent vertices in G_1 receive colors that differ by at least 2. In Section 4 we will discuss some of the existing results in this subarea, with an emphasis on recent results concerning a prelabeled version of this problem.

1.4 Backbone Coloring

The last type of coloring is the recently in [10] introduced notion of backbone coloring. In this variant one models the situation that the transmitters form a network in which a certain substructure of adjacent transmitters (called the backbone) is more crucial for the communication than the rest of the network. This means one should put more restrictions on the assignment of frequency channels along the backbone than on the assignment of frequency channels to other adjacent transmitters. The backbone could e.g. model so-called hot spots in the network where a very busy pattern of communications takes place, whereas the other adjacent transmitters supply a more moderate service. This leads to the problem of coloring the graph G_2 (that models the whole network) with a proper vertex coloring such that the colors on adjacent vertices in G_1 (that model the backbone) differ by at least 2. So far three types of backbones have been considered: (perfect) matchings, spanning trees and a special type of spanning trees also known as Hamiltonian paths. In Section 5 we will discuss the existing results and many open problems in this subarea. Note that the notion of backbone coloring in fact generalizes both radio coloring and radio labeling: radio coloring is the special case of backbone coloring in which G_1 is the backbone of $G_2 = G_1^2$, while radio labeling is the special case in which G_1 is the backbone of K_n.

2 Distant-2 Coloring

In this section we will survey some of the existing results on distant-2 coloring, and discuss the proof techniques and open problems in this subarea. We refer to [1], [7], [8], [29], [33], and [38] for more details.

2.1 Introduction and Main Results

Throughout Section 2, G is a plane graph (i.e., a representation in the plane of a planar graph), that is simple (i.e., without loops and multiple edges) and with vertex set V and edge set E. The *distance* between two vertices u and v is the length of a shortest path joining them.

A *distant-2-coloring* of G is a coloring of the vertices such that vertices at distance one or two have different colors. The least number for which a distant-2-coloring exists is called the *distant-2 chromatic number* of G, denoted by $\chi_2(G)$. We recall that a distant-2-coloring of G is equivalent to an ordinary vertex coloring of the square G^2 of G. (The *square* of a graph G, denoted G^2, is the graph with the same vertex set and in which two vertices are joined by an edge if and only if they have distance one or two in G.) And hence the distant-2 chromatic number $\chi_2(G)$ equals the ordinary *chromatic number* $\chi(G^2)$ of G^2.

The following conjecture was formulated in Wegner [38]. (See also Jensen & Toft [[30], Section 2.18].)

Conjecture 1. *If G is a planar graph with maximum degree Δ, then*

$$\chi_2(G) \le \begin{cases} \Delta + 5, & \textit{if } 4 \le \Delta \le 7; \\ \lfloor \frac{3}{2}\Delta \rfloor + 1, & \textit{if } \Delta \ge 8. \end{cases}$$

A first result towards a proof of this conjecture can be found in work of Jonas [31]. From one of the results in [31] it follows directly that $\chi_2(G) \leq 8\,\Delta - 22$ for a planar graph G with maximum degree $\Delta \geq 7$. This bound was significantly improved in Van den Heuvel & McGuinness [29] to $\chi_2(G) \leq 2\,\Delta + 25$. Independently, a result with a smaller factor in front of the Δ was proved by Agnarsson & Halldórsson [1] who showed that, provided $\Delta \geq 749$, for a planar graph G with maximum degree Δ we have $\chi_2(G) \leq \lfloor \frac{9}{5}\,\Delta \rfloor + 2$.

In [8], the lower bound on Δ for this last bound has been reduced. In fact, the following result was proved there.

Theorem 2. *If G is a planar graph with maximum degree Δ, then*

$$\chi_2(G) \leq \begin{cases} 59, & \text{if } \Delta \leq 20; \\ \max\{\,\Delta + 39,\ \lceil \frac{9}{5}\,\Delta \rceil + 1\,\}, & \text{if } \Delta \geq 21. \end{cases}$$

In particular, if $\Delta \geq 47$, then $\chi_2(G) \leq \lceil \frac{9}{5}\,\Delta \rceil + 1$.

The proof of Theorem 2 in [8] involves the establishment of the existence of certain *unavoidable configurations* in a planar graph, first of all in [7] for maximal planar graphs. The existence of these configurations is proved by the *discharging technique.* This approach goes back to Heawood's proof of the 5-Color Theorem [27], and the old and new proofs of the 4-Color Theorem ([3], [4], [35]).

To give the general idea, let us repeat the structure of the unavoidable configurations in [7] and [8]. They are defined in terms of "bunches" and "stars".

We say that G has a *bunch* of *length* $m \geq 3$ with as *poles* the vertices p and q, where $p \neq q$, if G contains a sequence of paths $Q_1, Q_2, \ldots, Q_m$ with the following properties. Each Q_i has length 1 or 2 and joins p with q. Furthermore, for each $i = 1, \ldots, m-1$, the cycle formed by Q_i and Q_{i+1} is not separating in G (i.e., has no vertex of G inside). Moreover, this sequence of paths is maximal in the sense that there is no path Q_0 (or Q_{m+1}) that could be added to the bunch, preserving the above properties. If the cycle bounded by Q_1 and Q_m separates G, then the internal vertices of Q_1 and Q_m (if they exist) are the *end vertices* of the bunch. A path $Q_i = pq$ of length 1 in the bunch will be referred to as a *parental edge.*

A *d-vertex* in G is a vertex of degree d. The *big vertices* in G are those of degree at least 26, and *minor vertices* those of degree at most 5.

Let u be a d-vertex, and let $v_1, \ldots, v_k$ be adjacent to u for some integer k with $1 \leq k \leq d$. We say that the vertices $u, v_1, \ldots, v_k$ and edges $uv_1, \ldots, uv_k$ form a *k-star at u*, defined by $v_1, \ldots, v_k$, of *weight* $\sum_{i=1}^{k} d(v_i)$. A $(d-1)$-star at a d-vertex is called *precomplete*, and a d-star at a d-vertex is *complete.*

The following result describes the unavoidable configurations used in [8] to prove Theorem 2 on distant-2-colorings. We omit the proof.

Theorem 3. *For each plane graph G at least one of the following holds:*

(a) G has a precomplete star of weight at most 38 that does not contain big vertices and is centred at a minor vertex.

(b) G has a big vertex b that satisfies at least one of the following conditions:

- *b is a pole for a bunch of length greater than $d(b)/5$;*
- *b is a pole for a bunch of length precisely $d(b)/5$ with a parental edge;*
- *b is a pole for 5 bunches of length $d(b)/5$ without parental edges and with pairwise different end vertices. Moreover, among the end vertices there is a vertex v_0 of degree at most 11, and each other end vertex has degree at most 5. Furthermore, if v_i and v_{i+1} are consecutive in the vicinity of b and are end vertices of two bunches such that $v_i \neq v_0$ and $d(v_i) = 5$, then v_i and v_{i+1} are adjacent in G.*

As proved in Borodin & Woodall [9], each plane graph with minimum degree 5 has a precomplete star of weight at most 25 centred at a 5-vertex. On the other hand, planar graphs with vertices of degree less than 5 may have arbitrarily large weight of the precomplete stars at all minor vertices, as follows from the n-bipyramid. Theorem 3 shows that this is only possible if there are long enough bunches at big vertices.

This structural result of Theorem 3 can be used to prove Theorem 2 by induction (contracting an edge in a suitable star or bunch). In fact, the structural result is used to first prove a best possible upper bound on the minimum degree of the square of a planar graph, and hence on a best possible bound for the number of colors needed in a greedy coloring of it.

Using different unavoidable configurations and a proof which is also based on the discharging method, Molloy & Salavatipour were able to prove the following considerable strengthening of Theorem 2 in [33].

Theorem 4. *If G is a planar graph with maximum degree Δ, then*

$$\chi_2(G) \le \begin{cases} \lfloor \frac{5}{3}\Delta \rfloor + 78, & \\ \lfloor \frac{5}{3}\Delta \rfloor + 24, & \text{if } \Delta \ge 241. \end{cases}$$

Based on the length and depth of the proofs of the previous results one is likely to think that further improvements should be based on a different proof approach, avoiding complicated discharging and unavoidable configurations.

Recently, Andreou & Spirakis announced an "almost proof" of Conjecture 1 in [2].

3 Radio Coloring

In some versions of the previous problem one puts the additional restriction on G_1 that the colors should be sufficiently separated, in order to model practical frequency assignment problems in which interference should be kept at an acceptable level. One way to model this is to use positive integers for the colors (modeling certain frequency channels) and to ask for a coloring of G_1 and

$G_2 = G_1^2$ such that the colors on adjacent vertices in G_2 are different, whereas they differ by at least 2 on adjacent vertices in G_1. This problem is known as the radio coloring problem and has been studied (under various names, e.g. $L(2,1)$-labeling, $\lambda_{2,1}$-coloring and $\chi_{2,1}$-labeling) in [6], [12], [15], [16], [17], [18], and [31].

A *radio coloring* of a graph $G = (V, E)$ is a function $f: V \to \mathbf{N}^+$ such that $|f(u) - f(v)| \geq 2$ if $\{u, v\} \in E$ and $|f(u) - f(v)| \geq 1$ if the distance between u and v in G is 2. The notion of radio coloring was introduced by GRIGGS & YEH [23] under the name $L(2,1)$-labeling. The *span* of a radio coloring f of G is $\max_{v \in V} f(v)$.

The problem of determining a radio coloring with minimum span has received a lot of attention. For various graph classes the problem was studied by SAKAI [36], BODLAENDER, KLOKS, TAN & VAN LEEUWEN [6], VAN DEN HEUVEL, LEESE & SHEPHERD [28], and others. NP-hardness results for this RADIO COLORING problem (RC) restricted to planar, split, or cobipartite graphs were obtained by BODLAENDER, KLOKS, TAN & VAN LEEUWEN [6]. Fixed-parameter tractability properties of RC are discussed by FIALA, KRATOCHVÍL AND KLOKS [16]. FIALA, FISHKIN & FOMIN [15] study on-line algorithms for RC. For only very few graph classes the problem is known to be polynomially solvable. CHANG & KUO [12] obtained a polynomial time algorithm for RC restricted to trees and cographs. The complexity of RC even for graphs of treewidth 2 is a long standing open question. An interesting direction of research was initiated by FIALA, KRATOCHVÍL & PROSKUROWSKI [17]. They consider a precolored version of RC, i.e. a version in which some colors are pre-assigned to some vertices. They proved that RC with a given precoloring can be solved in polynomial time for trees. Recently GOLOVACH [21] proved that RC is NP-hard for graphs of treewidth 2.

Due to page limitations we omit the details and confine ourselves to referring the interested reader to the cited papers.

4 Radio Labeling

The so-called radio labeling problem models a practical setting in which all assigned frequency channels should be distinct, with the additional restriction that adjacent transmitters should use sufficiently separated frequency channels. Within the above framework this can be modeled by considering the graph G_1 that models the adjacencies of n transmitters, and taking $G_2 = K_n$, the complete graph on n vertices. The restrictions are clear: one asks for a proper vertex coloring of G_2 such that adjacent vertices in G_1 receive colors that differ by at least 2.

4.1 Definitions and Preliminary Observations

The *girth* of a graph G is the length of a shortest cycle in G. A graph G is *t-degenerate* if each of its subgraphs has a vertex of degree at most t. A *labeling* of the (vertex set of the) graph $G = (V, E)$ is an injective mapping $L: V \to \mathbf{N}^+$. A labeling L of G is called a *radio labeling* of G if for any edge $\{u, v\} \in E$ the inequality $|L(u) - L(v)| \geq 2$ holds; the *span* of such a labeling L is $\max_{v \in V} L(v)$.

The Radio Labeling problem (RL) is defined as follows: "For a given graph G, find a radio labeling L with the smallest span." The name radio labeling was suggested by Fotakis & Spirakis in [19] but the same notion (under different names) has been introduced independently and earlier by other researchers (see, e.g. Chang & Kuo [12]). Problem RL is equivalent to the special case of the Traveling Salesman problem TSP(2,1) in which all edge weights (distances) are either one or two. The relation is as follows. For a graph $G = (V, E)$ let K_G be the complete weighted graph on V with edge weights 1 and 2 defined according to E: for every $\{u, v\} \in E$ the weight $w(\{u, v\})$ in K_G is 2 and for $\{u, v\} \notin E$ the weight $w(\{u, v\}) = 1$. The *weight* of a path in K_G is the sum of the weights of its edges. The following proposition can be found in [19, 18].

Proposition 5. *There is a radio labeling of G with span k if and only if there is a Hamiltonian path (i.e. a path on $|V|$ vertices) of weight $k-1$ in K_G.*

Another equivalent formulation of this problem, which was extensively studied in the literature, is the Hamiltonian Path Completion problem (HPC), i.e. the problem of partitioning the vertex set of a graph G into the smallest possible number of sets which are spanned by paths in G. This equivalence is expressed in the following well-known proposition. Here $\overline{G}$ denotes the *complement* of G, i.e. the graph obtained from a complete graph on $|V(G)|$ vertices by deleting the edges of G.

Proposition 6. *There is a radio labeling of G with span $\leq k$ if and only if there is a partition of V into $\leq k$ sets, such that each of these sets induces a subgraph in $\overline{G}$ that contains a Hamiltonian path.*

As we mentioned above the Traveling Salesman problem TSP(2,1) (which is equivalent to RL) is a well-studied problem. Papadimitriou & Yannakakis [34] proved that this problem is MAX SNP-hard, but gave an approximation algorithm for TSP(2,1) which finds a solution not worse than 7/6 times the optimum solution. Later Engebretsen [14] improved their result by showing that the problem is not approximable within $5381/5380 - \varepsilon$ for any $\varepsilon > 0$.

Damaschke, Deogun, Kratsch & Steiner [13] proved that the Hamiltonian Path Completion problem HPC can be solved in polynomial time on cocomparability graphs (complements of comparability graphs). To obtain this result they used a reduction to the problem of finding the bump number of a partial order. (The *bump number* of a poset P and its linear extension L is the number of neighbors in L which are comparable in P.) It was proved by Habib, Möhring & Steiner [24] and by Schäffer & Simons [37] that the Bump Number problem can be solved in polynomial time. By Proposition 6, the result of Damaschke, Deogun, Kratsch & Steiner yields that RL is polynomial time solvable for comparability graphs. Later, this result was rediscovered by Chang & Kuo [12] but under the name of $L'(2,1)$-labeling and only for cographs, a subclass of the class of comparability graphs. Notice that RL is NP-hard for cocomparability graphs because the Hamiltonian Path problem is known to be NP-hard for bipartite graphs which form a subclass of comparability graphs. Recently, Fotakis & Spirakis [19] proved that RL can be solved in polynomial time within the class of graphs for which a k-coloring can be obtained in polynomial time (for some fixed k). Note that, for example, this class of graphs includes the well-studied classes of planar graphs and graphs with bounded treewidth.

4.2 Radio Labeling with Prelabeling

Here we focus briefly on a recently studied prelabeling version of the radio labeling problem ([5]).

For a graph $G = (V, E)$ a *pre-labeling* L' of a subset $V' \subset V$ is an injective mapping $L' \colon V' \to \mathbf{N}^+$. We say that a labeling L of G *extends* the pre-labeling L' if $L(u) = L'(u)$ for every $u \in V'$. We consider the following two problems that were introduced in [5]:

– p-RL($*$): RL with an arbitrary number of pre-labeled vertices.
  For a given graph G and a given pre-labeling L' of G, determine a radio labeling of G extending L' with the smallest span.
– p-RL(l): RL with a fixed number of pre-labeled vertices.
  For a given graph $G = (V, E)$, a subset $V' \subseteq V$ with $|V| \leq l$, and a pre-labeling $L' \colon V' \to \mathbf{N}^+$, determine a radio labeling of G extending L' with the smallest span.

In [5], the authors studied algorithmical, complexity-theoretical, and combinatorial aspects of radio labeling with pre-labeled vertices. We will briefly summarize these results in Section 4.3, and list some open problems in Section 4.4.

4.3 Upper Bounds for the Minimum Span

Let $G = (V, E)$ denote a graph on n vertices, and let $V' \subseteq V$ and $L' \colon V' \to \mathbf{N}^+$ be a fixed subset of V and a pre-labeling for V', respectively. Let

$$M \ := \ \max\{n, \ \max_{v \in V'} L'(v)\}.$$

Clearly, M is straightforward to compute if G and L' are known. And clearly, M is a lower bound on the span of any radio labeling in G extending the pre-labeling L' of G. A natural question is how far M can be away from the minimum span of such a labeling. In [5] it is shown that the answer to this question heavily relies on the girth of the graph G:

Theorem 7. *Consider a graph G on $n \geq 7$ vertices, and a pre-labeling L' of G. Then there is a radio labeling in G extending L'*

(a) with span $\leq \lfloor (7M - 2)/3 \rfloor$
(b) with span $\leq \lfloor (5M + 2)/3 \rfloor$ if G has girth at least 4
(c) with span $\leq M + 3$ if G has girth at least 5.

All these bounds are best possible. The third bound is even best possible for the class of paths.

For graphs with bounded degeneracy the following results are proved in [5].

Lemma 8 *If G is a t-degenerate graph on n vertices, then it has a radio labeling with span $\leq n + 2t$.*

Theorem 9. *If G is a t-degenerate graph and L' is a pre-labeling of G, then there exists a radio labeling extending L' with span $\leq M + (4 + \sqrt{3})t + 1$.*

The above results imply a polynomial time approximation algorithm for solving the radio labeling problem (with pre-labeling) in t-degenerate graphs. The bound in Theorem 9 can possibly be improved considerably. In [5] the following conjecture is posed.

Conjecture 10. *If G is a t-degenerate graph and L' is a pre-labeling of G, then there exists a radio labeling extending L' with span $\leq M + 3t$.*

The upper bound in the above conjecture cannot be improved.

We now turn to graphs with a bounded maximum degree.

Theorem 11. *Let $G = (V, E)$ be a t-degenerate graph with maximum degree Δ and let $V' \subseteq V$ be the set of vertices that is pre-labeled by L'. If the number of unlabeled vertices $p = |V \setminus V'| \geq 4\Delta(t + 1)$, then L' can be extended to a radio labeling of G with span M.*

In [5] Theorem 11 is used to obtain the following complexity result for graphs with a bounded maximum degree.

Corollary 12. *Let k be a fixed positive integer. For every graph G with maximum degree $\Delta \leq k$ and pre-labeling L',* P-RL(∗) *can be solved in polynomial time.*

The above corollary shows that P-RL(l) and P-RL(∗) have the same complexity behavior as RL for graphs with a bounded maximum degree, i.e. all three of the problems can be solved in polynomial time. This picture changes if we restrict ourselves to graphs which are k-colorable and for which a k-coloring is given (as part of the input) for some fixed positive integer k.

Related to Proposition 5 we discussed the useful equivalence between RL and the TRAVELING SALESMAN problem TSP(2,1). In [5] this equivalence is adapted as follows to capture the restrictions of the pre-labeling problem. Let L be a labeling of a graph $G = (V, E)$ on n vertices. The path $P = (v_1, v_2, \ldots, v_n)$ *corresponding* to L visits the vertices by increasing labels, i.e. for all $1 \leq a < b \leq n$ we have $L(v_a) < L(v_b)$. P is a path in the complete graph K_G; its *weight* $w(P)$ is measured according to the edge weights w in K_G. In [5] the following result is proved and used to obtain the next corollary.

Theorem 13. *Let $G = (V, E)$ be a graph with a given k-coloring with color classes $I_1, I_2, \ldots, I_k$. Let L' be a pre-labeling of a subset $V' \subseteq V$ with $|V'| = l$. Then a radio labeling L of G extending L' with the smallest possible span can be computed in time $O(n^{4(l+1)k(k-1)})$.*

For each of the graph classes in the following corollary, it is possible to construct a vertex coloring with a constant number of colors in polynomial time. Hence:

Corollary 14. *The radio labeling problem* P-RL(l) *is polynomially solvable*

- *on the class of planar graphs,*
- *on any class of graphs of bounded treewidth,*
- *on the class of bipartite graphs.*

The above results show that P-RL(l) is solvable in polynomial time for graphs with a bounded chromatic number and a given coloring. As shown in [5] this result does not carry over to the more general labeling problem P-RL($*$) where the number of pre-labeled vertices is part of the input. It is shown there that P-RL($*$) is NP-hard even when restricted to 3-colorable graphs with a given 3-coloring by a transformation from PARTITION INTO TRIANGLES; this result is then easily generalized in [5] to k-colorable graphs ($k \geq 4$) with a given k-coloring.

Theorem 15. *For any fixed $k \geq 3$, problem* P-RL($*$) *is NP-hard even when the input is restricted to graphs with a given k-coloring.*

The final results in [5] refer to another class of graphs for which RL is known to be polynomially solvable, namely the class of cographs, i.e. graphs without an induced path on four vertices. Using an easy reduction from 3-PARTITION it is shown in [5] that P-RL($*$) is NP-hard for cographs.

The complexity of P-RL(l) for cographs is left in [5] as one of the open problems.

4.4 Open Problems

In this section we focussed on two versions of the radio labeling problem in which a pre-labeling is assumed. The known results are summarized in the following table.

	graphs with a bounded Δ	graphs with a given k-coloring	Cographs
RL	P [19]	P [19]	P [13, 12]
P-RL(l)	P [5]	P [5]	???
P-RL($*$)	P [5]	NP for $k \geq 3$ [5]	NP [5]

In this table, an entry P denotes solvable in polynomial time, NP denotes NP-hard, and the sign ??? marks an open problem.

For the results in the middle column, we assume that k is a fixed integer that is not part of the input. Note that the class of graphs with a given k-coloring contains important and well-studied graph classes such as the class of planar graphs and the class of graphs with bounded treewidth.

Many questions remain open, a few of which are listed in [5] and repeated below:

– The complexity of any of the variants of Radio Labeling (RL, p-RL(l) and p-RL($*$)) for interval graphs are open problems.
– As mentioned earlier another open problem concerns the computational complexity of p-RL(l) for cographs.
– The results in [5] imply that p-RL(l) is polynomial for bipartite graphs. On the other hand, it is proved there that p-RL($*$) is NP-hard for 3-partite graphs even if a 3-coloring of the graph is given. The complexity of p-RL($*$) for bipartite graphs is open.
– As shown in [5] p-RL(l) is polynomial for planar graphs and graphs of bounded treewidth. The complexity of p-RL($*$) for these graph classes is open.

5 Backbone Coloring

5.1 Introduction and Terminology

In this last section we consider backbone colorings, a variation on classical vertex colorings that was introduced in [10]: Given a graph $G = (V, E)$ and a spanning subgraph H of G (the backbone of G), a *backbone coloring* for G and H is a proper vertex coloring $V \to \{1, 2, \ldots\}$ of G in which the colors assigned to adjacent vertices in H differ by at least two. We recall that the chromatic number $\chi(G)$ is the smallest integer k for which there exists a proper coloring $f : V \to \{1, \ldots, k\}$. The *backbone coloring number* $\text{BBC}(G, H)$ of (G, H) is the smallest integer ℓ for which there exists a backbone coloring $f : V \to \{1, \ldots, \ell\}$.

In [10] the results are concentrated on cases where the backbone is either a spanning tree or a spanning path, in [11] the backbone is a perfect matching. In both [10] and [11] combinatorial and algorithmic aspects are treated. We summarize the main results from [10] and [11] in the next two subsections, but first introduce some additional terminology and notation.

A *Hamiltonian path* of the graph $G = (V, E)$ is a path containing all vertices of G, i.e. a sequence $(v_1, v_2, \ldots, v_n)$ such that $V = \{v_1, v_2, \ldots, v_n\}$, all v_i are distinct, and $\{v_i, v_{i+1}\} \in E$ for all $i = 1, 2, \ldots, n-1$. A *perfect matching* is a subset of $|V|/2$ edges from E in which none of the edges share a common end vertex. A *split graph* is a graph whose vertex set can be partitioned into a *clique* (i.e. a set of mutually adjacent vertices) and an *independent set* (i.e. a set of mutually nonadjacent vertices), with possibly edges in between. The size of a largest clique in G is denoted by $\omega(G)$. Split graphs are perfect graphs, and hence satisfy $\chi(G) = \omega(G)$.

5.2 Relations with the Chromatic Number

Part of the results in [10] and [11] are motivated by the following question: How far away from $\chi(G)$ can $\text{BBC}(G, H)$ be in the worst case? To answer this question, in [10] the authors introduced, for integers $k \geq 1$, the values

$$\mathcal{T}(k) = \max\{\text{BBC}(G, T) : \ G \text{ with spanning tree } T, \text{ and } \chi(G) = k\} \qquad (1)$$

As shown in [10], it turns out that this function $\mathcal{T}(k)$ behaves quite primitively:

Theorem 16. *$\mathcal{T}(k) = 2k - 1$ for all $k \geq 1$.*

The upper bound $\mathcal{T}(k) \leq 2k - 1$ in this theorem in fact is straightforward to see. Indeed, consider a proper coloring of G with colors $1, \ldots, \chi(G)$, and replace every color i by a new color $2i - 1$. The resulting coloring uses only odd colors, and hence constitutes a 'universal' backbone coloring for any spanning tree T of G. The proof (in [10]) of the matching lower bound $\mathcal{T}(k) \geq 2k - 1$ is more involved and is omitted.

Next, let us discuss the situation where the backbone tree is a Hamiltonian path. Similarly as in (1), in [10] the authors introduced, for integers $k \geq 1$, the values

$$\mathcal{P}(k) = \max\{\text{BBC}(G, P):\ G \text{ with Hamiltonian path } P, \text{ and } \chi(G) = k\} \quad (2)$$

In [10] all the values of $\mathcal{P}(k)$ were exactly determined. They roughly grow like $3k/2$. Their precise behavior is summarized in the following theorem.

Theorem 17. *For $k \geq 1$ the function $\mathcal{P}(k)$ takes the following values:*

(a) For $1 \leq k \leq 4$: $\mathcal{P}(k) = 2k - 1$;
(b) $\mathcal{P}(5) = 8$ and $\mathcal{P}(6) = 10$;
(c) For $k \geq 7$ and $k = 4t$: $\mathcal{P}(4t) = 6t$;
(d) For $k \geq 7$ and $k = 4t + 1$: $\mathcal{P}(4t + 1) = 6t + 1$;
(e) For $k \geq 7$ and $k = 4t + 2$: $\mathcal{P}(4t + 2) = 6t + 3$;
(f) For $k \geq 7$ and $k = 4t + 3$: $\mathcal{P}(4t + 3) = 6t + 5$;

In [10] the authors also discuss the special case of backbone colorings on split graphs. Split graphs were introduced by Hammer & Földes [26]; see also the book [22] by Golumbic. They form an interesting subclass of the class of perfect graphs. The combinatorics of most graph problems becomes easier when the problem is restricted to split graphs. The following theorem from [10] is a strengthening of Theorems 16 and 17 for the special case of split graphs.

Theorem 18. *Let $G = (V, E)$ be a split graph.*

(a) In G is connected, then for every spanning tree T in G, $\text{BBC}(G, T) \leq \chi(G) + 2$.
(b) If $\omega(G) \neq 3$ and G contains a Hamiltonian path, then for every Hamiltonian path P in G, $\text{BBC}(G, P) \leq \chi(G) + 1$.

Both bounds are tight.

An example in [10] shows why for split graphs with clique number 3 the statement in Theorem 18(b) does not work.

In [11], similar as in (1) and (2), for integers $k \geq 1$, the values

$$\mathcal{M}(k) = \max\{\text{BBC}(G, M):\ G \text{ with perfect matching } M, \text{ and } \chi(G) = k\} \quad (3)$$

were introduced and exactly determined. These values roughly grow like $4k/3$. Their precise behavior is summarized in the following theorem,

Theorem 19. *For $k \geq 1$ the function $\mathcal{M}(k)$ takes the following values:*

(a) $\mathcal{M}(4) = 6$*;*
(b) For $k = 3t$*:* $\mathcal{M}(3t) = 4t$*;*
(c) For $k \neq 4$ *and* $k = 3t + 1$*:* $\mathcal{M}(3t + 1) = 4t + 1$*;*
(d) For $k = 3t + 2$*:* $\mathcal{M}(3t + 2) = 4t + 3$*.*

5.3 Complexity of Backbone Coloring

In [10] and [11] the authors also discussed the computational complexity of computing the backbone coloring number: *"Given a graph G, a spanning subgraph H, and an integer ℓ, is* $\textsc{bbc}(G, H) \leq \ell$*?"* Of course, this general problem is NP-complete. It turns out that in case H is a spanning tree for this problem the complexity jump occurs between $\ell = 4$ (easy for all spanning trees) and $\ell = 5$ (difficult even for Hamiltonian paths). This is proved in [10].

Theorem 20.

(a) The following problem is polynomially solvable for any $\ell \leq 4$: Given a graph G and a spanning tree T of G, decide whether $\textsc{bbc}(G, T) \leq \ell$*.*
(b) The following problem is NP-complete for all $\ell \geq 5$: Given a graph G and a Hamiltonian path P of G, decide whether $\textsc{bbc}(G, P) \leq \ell$*.*

As shown in [11], in case H is a perfect matching, the complexity jump occurs between $\ell = 3$ and $\ell = 4$.

Theorem 21.

(a) The following problem is polynomially solvable for any $\ell \leq 3$: Given a graph G and a perfect matching M of G, decide whether $\textsc{bbc}(G, M) \leq \ell$*.*
(b) The following problem is NP-complete for all $\ell \geq 4$: Given a graph G and a perfect matching M of G, decide whether $\textsc{bbc}(G, M) \leq \ell$*.*

5.4 Discussion

In [10] and [11] the combinatorics and the complexity of backbone colorings of graphs have been analyzed, where the backbone is formed by a Hamiltonian path, by a spanning tree, or by a matching.

Since this area is new, it contains many open problems. For *arbitrary* graphs G with spanning tree T, the backbone coloring number $\textsc{bbc}(G, T)$ can be as large as $2\chi(G) - 1$. What about *triangle-free* graphs G? Does there exist a small constant c such that $\textsc{bbc}(G, T) \leq \chi(G) + c$ holds for all triangle-free graphs G? And what about *chordal* graphs? It can be shown that $\textsc{bbc}(G, P) \leq \chi(G) + 4$ whenever G is chordal and P is a Hamiltonian path of G. Does this result carry over to arbitrary spanning trees, i.e., does $\textsc{bbc}(G, T) \leq \chi(G) + c$ hold for any chordal graph G with spanning tree T?

Finally, what about *planar* graphs? The 4-Color Theorem together with Theorem 16 implies that $\textsc{bbc}(G, T) \leq 7$ holds for any planar graph G with spanning tree T. However, this bound 7 is probably not best possible. Can it be improved to 6? There are planar graphs that demonstrate that this bound can not be improved to 5, even for Hamiltonian path backbones. What about perfect matching backbones? The 4-Color Theorem together with Theorem 19 implies that $\text{BBC}(G, M) \leq 6$ holds for any planar graph G with perfect matching M. It seems that this bound 6 is not best possible, but there are planar graphs showing that we cannot improve this bound to 4.

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Crossing Numbers and Skewness of Some Generalized Petersen Graphs

G.L. Chia and C.L. Lee

Institute of Mathematical Sciences, University of Malaya,
50603 Kuala Lumpur, Malaysia

Abstract. The *skewness* of a graph G is the minimum number of edges in G whose removal results in a planar graph. In this paper, we show that the skewness of the generalized Petersen graph $P(3k, k)$ is $\lceil \frac{k}{2} \rceil + 1$, where $k \geq 4$. As a byproduct, it is shown that for $k \geq 4$, $\lceil \frac{k}{2} \rceil + 1 \leq cr(P(3k, k)) \leq k$, where $cr(G)$ denotes the crossing number of G.

Let G be a graph. The *crossing number* of G, denoted $cr(G)$, is the minimum number of pairwise intersections of its edges when G is drawn in the plane. The problem of determining the crossing number of a given graph has been shown to be NP-complete (see [6]).

Let n and k be two integers such that $1 \leq k \leq n-1$. The *generalized Petersen graph* $P(n, k)$ is defined to have vertex-set $\{u_i, v_i : i = 0, 1, \ldots, n-1\}$ and edge-set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 0, 1, \ldots, n-1$ with subscripts reduced modulo $n\}$.

A number of papers (see [4], [5], [8], [9] and [11]) have gone into finding the crossing numbers of some generalized Petersen graphs. For the general situation, only lower and upper bounds are known for $P(n, k)$ (see [10]). In this paper, we attempt to look at the crossing number of $P(3k, k)$.

Definition 1. *Let G be a graph. The skewness of G, denoted $sk(G)$, is defined to be the minimum number of edges in G whose removal results in a planar graph. If G has skewness r, let $R(G)$ denote a set of r edges in G whose removal results in a planar graph.*

The concept of skewness of graph has been previously defined and exploited in [3] and [7]. Unaware of it, this number has been called the *removal number* (and denoted $rem(G)$) in [1] and [2].

It is clear that, for any graph G, $cr(G) \geq sk(G)$. In this paper, we determine the skewness for $P(3k, k)$ and hence a lower bound for $cr(P(3k, k))$. In addition, an upper bound for $cr(P(3k, k))$ is also given (see Corollary 2).

Let $\lceil x \rceil$ denote the least integer greater or equal to x.

Theorem 1. *The skewness of $P(3k, k)$ is $\lceil \frac{k}{2} \rceil + 1$, where $k \geq 4$.*

Note that the graph $P(3k, k)$ contains k triangles. A careful check shows that replacing each of these triangles by a vertex of degree 3 does not affect

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the crossing number or the skewness of $P(3k, k)$. (Note that this fact is not true in general.) Let $Q(k)$ denote the resulting graph obtained from $P(3k, k)$ by replacing each triangle by a vertex of degree 3. A formal definition of the graph $Q(k)$ is given below. Therefore, we shall prove Theorem 1 by showing that the skewness of $Q(k)$ is $\lceil \frac{k}{2} \rceil + 1$ where $k \geq 4$.

Definition 2. *Let $k \geq 1$ be an integer. The graph $Q(k)$ is defined to have vertex-set $\{0, 1, \ldots, 3k-1, x_0, x_1, \ldots, x_{k-1}\}$ and edge-set $\{i(i+1), jx_j, (j+k)x_j, (j+2k)x_j : i = 0, 1, \ldots, 3k-1,\ j = 0, 1, \ldots, k-1$ with the operations reduced modulo $3k$ and those on the subscripts reduced modulo $k\}$. Edges of the form $i(i+1)$ are called rim edges while those of the form rx_j are called spokes. The vertex x_j is called an internal vertex of $Q(k)$ while each vertex i is called an external vertex.*

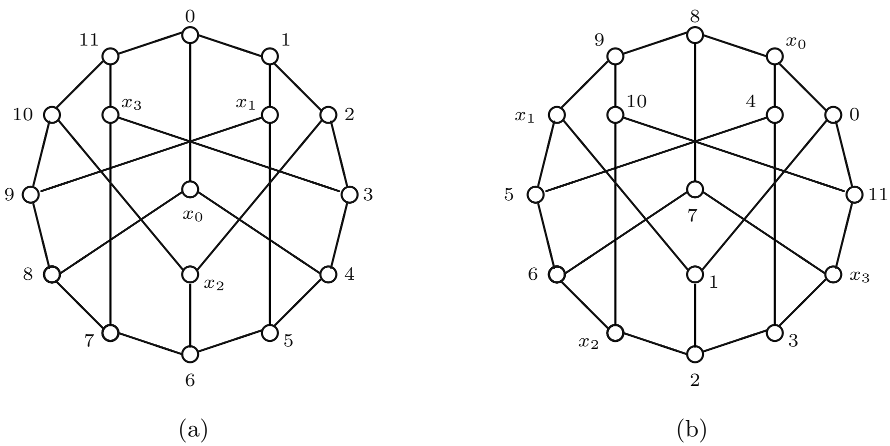


Fig. 1. The graph $Q(4)$

Lemma 1. *$Q(4)$ is both vertex and edge transitive.*

Proof. First, we note any two external (respectively internal) vertices of $Q(4)$ are similar. Next, any two rim edges (respectively spokes) of $Q(4)$ are similar.

The graph $Q(4)$ is shown in Fig. 1(a). With the same labels on its vertices, this graph is redrawn as shown in Fig. 1(b). These drawings imply that each external vertex is similar to each internal vertex and that each rim edge is similar to each spoke. Therefore $Q(4)$ is both vertex and edge transitive. □

Lemma 2. *Suppose $sk(Q(k)) = sk(Q(k-1))$, $k \geq 4$. Then $R(Q(k))$ is a union of independent rim edges of $Q(k)$.*

Proof. Suppose $sk(Q(k)) = sk(Q(k-1))$. First, we show that

(i) $R(Q(k))$ contains no spokes of $Q(k)$.

Let f be a spoke of $Q(k)$ and let H_0 denote the subgraph of $Q(k)$ obtained by deleting the edge f. If $f \in R(Q(k))$, then $sk(H_0) = sk(Q(k)) - 1$. Since H_0

contains a subdivision of $Q(k-1)$ as subgraph, we have $sk(H_0) \geq sk(Q(k-1))$. Consequently, we have $sk(Q(k)) \geq sk(Q(k-1)) + 1$, a contradiction.

Next, we show that

(ii) no adjacent rim edges of $Q(k)$ are contained in $R(Q(k))$.

Let e be a rim edge of $Q(k-1)$ and let H_1 denote the subgraph of $Q(k-1)$ obtained by deleting the edge e. Then clearly $sk(H_1) \geq sk(Q(k-1)) - 1$.

Now, let e_1 and e_2 be two adjacent rim edges of $Q(k)$ and assume that $e_1, e_2 \in R(Q(k))$. Let M denote the subgraph of $Q(k)$ obtained by deleting the edges e_1 and e_2. Then $sk(M) = sk(Q(k)) - 2$. Since M contains a subdivision of H_1 as subgraph, we have $sk(M) \geq sk(H_1)$. Consequently, we have $sk(Q(k)) \geq sk(Q(k-1)) + 1$, a contradiction. □

Lemma 3. *$sk(Q(3)) = 2$.*

Proof. In [5], it is shown that $cr(P(9,3)) = 2$ and so $sk(Q(3)) \leq 2$ (since $cr(Q(3)) = cr(P(9,3))$).

It is routine to verify that the resulting graph obtained from $Q(3)$ by deleting any external vertex, together with all edges incident to it, is non-planar. This implies that if any edge, whether rim edge or spoke, is deleted from $Q(3)$, the resulting is non-planar. This implies that $sk(Q(3)) \geq 2$. Hence we have $sk(Q(3)) = 2$ □

Corollary 1. *$sk(Q(4)) \geq 3$.*

Proof. Note that the graph obtained from $Q(4)$ by deleting an internal vertex, together with all edges incident to it, is a subdivision of $Q(3)$. This implies that $sk(Q(4)) \geq sk(Q(3))$. Suppose $sk(Q(4)) = 2$. Since $sk(Q(3)) = 2$ by Lemma 3, we see that $R(Q(4))$ contains no spoke of $Q(4)$ by Lemma 2. But then, this is a contradiction because $Q(4)$ is edge transitive by Lemma 1. □

Suppose G is a graph and x is vertex in G. Let $N(x)$ denote the neighborhood of x. Further, let $N[x] := N(x) \cup \{x\}$.

Lemma 4. *$\lceil \frac{k}{2} \rceil \leq sk(Q(k)) \leq \lceil \frac{k}{2} \rceil + 1$ for any $k \geq 3$.*

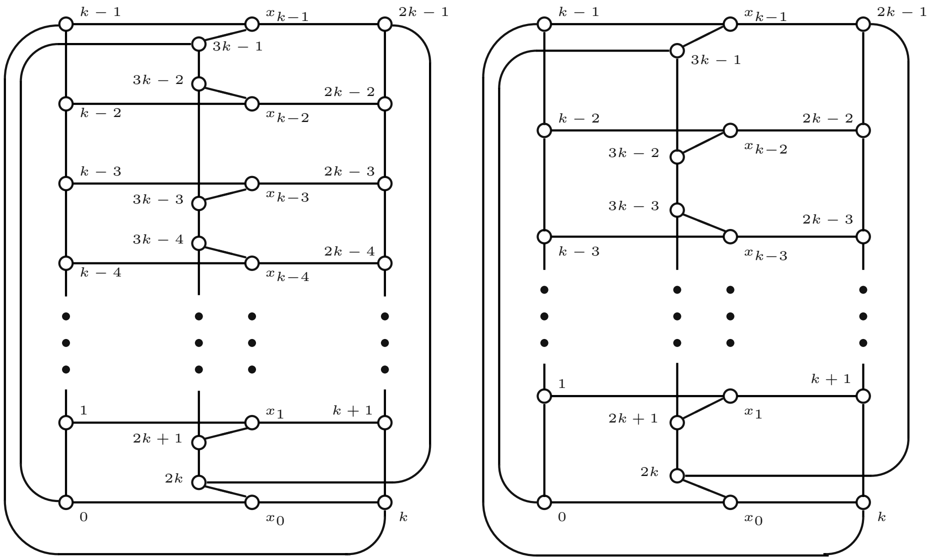
Proof. The two drawings of $Q(k)$ in Fig. 2 ((a) if k is even and (b) if k is odd) establish the upper bound. Note that, if k is even (respectively odd), then a planar graph can be obtained by deleting the set of $\frac{k+2}{2}$ (respectively $\frac{k+3}{2}$) edges given by $(2k+2i-1, 2k+2i)$, $i = 0, 1, \ldots, \frac{k-2}{2}$ (respectively $i = 0, 1, \ldots, \frac{k-1}{2}$) and $(0, 3k-1)$.

We prove the lower bound by induction on k. The lemma is true for $k \leq 4$ because $sk(Q(3)) = 2$ and $sk(Q(4)) \geq 3$ by Corollary 1.

Assume that $sk(Q(m)) \geq \lceil \frac{m}{2} \rceil$ for all $m \leq k-1$ where $k \geq 5$.

Clearly, $sk(Q(k)) \geq sk(Q(k-1))$. As such, it follows from the induction hypothesis that

$$sk(Q(k)) \geq \left\lceil \frac{k-1}{2} \right\rceil \tag{1}$$



(a) k is even.

(b) k is odd.

Fig. 2. Two drawings of the graph $Q(k)$, $k \geq 3$

If k is even, then $\lceil \frac{k-1}{2} \rceil = \lceil \frac{k}{2} \rceil$ and the lemma true in this case.
If k is odd, then $\lceil \frac{k-1}{2} \rceil = \lceil \frac{k}{2} \rceil - 1$ and we have

$$sk(Q(k)) \geq \left\lceil \frac{k}{2} \right\rceil - 1.$$

If equality does not hold for the above, then $sk(Q(k)) \geq \lceil \frac{k}{2} \rceil$ and the lemma follows by induction.

On the other hand, if equality holds, then it follows from the induction hypothesis that $sk(Q(k)) = sk(Q(k-1))$. By Lemma 2, $R(Q(k))$ contains only non-adjacent rim edges of $Q(k)$. We shall show that this case leads to a contradiction.

Clearly, any external vertex of $Q(k)$ is adjacent to an internal vertex of $Q(k)$. Since $sk(Q(k)) = \lceil \frac{k}{2} \rceil - 1$, there are at most $k-1$ internal vertices of $Q(k)$ whose neighbors are end vertices of some edges in $R(Q(k))$. This means that there is an internal vertex of $Q(k)$ whose neighbors are not end vertices of any edges in $R(Q(k))$. Without loss of generality, let this vertex be x_0.

Let H denote the subgraph of $Q(k)$ induced by the set of vertices in $N[x_{-1}] \cup N[x_0] \cup N[x_1]$. Then H is a subdivision of $K_{3,3}$. Moreover, H is a subgraph of $Q(k) - R(Q(k))$ because H contains no edges of $R(Q(k))$. But this is a contradiction because $Q(k) - R(Q(k))$ is planar. $\square$

Corollary 2. $\lceil \frac{k}{2} \rceil + 1 \leq cr(P(3k, k)) \leq k$ *for* $k \geq 4$.

Proof. The lower bound follows from Theorem 1 (because $cr(Q(k)) = cr(P(3k,k)))$. The two drawings of the graph $Q(k)$ in Fig. 2 show that $cr(Q(k)) \leq k$ for $k \geq 4$ and hence $cr(P(3k,k)) \leq k$. □

It is quite likely that $cr(P(3k,k)) = k$ for $k \geq 4$. We hope to report more about this in the near future.

The next lemma sets up the inductive step for the proof of Theorem 1 by induction.

Lemma 5. *Suppose $k \geq 5$ and $sk(Q(m)) = \lceil \frac{m}{2} \rceil + 1$ for all $m \leq k-1$. Then*

$$sk(Q(k)) = \left\lceil \frac{k}{2} \right\rceil + 1.$$

Proof. By assumption, we have $sk(Q(k-1)) = \lceil \frac{k-1}{2} \rceil + 1$. By Lemma 4,

$$\left\lceil \frac{k}{2} \right\rceil \leq sk(Q(k)) \leq \left\lceil \frac{k}{2} \right\rceil + 1.$$

Clearly, $sk(Q(k)) \geq sk(Q(k-1))$. As such, if k is even, then $\lceil \frac{k}{2} \rceil = \lceil \frac{k-1}{2} \rceil$ and we have, $sk(Q(k)) = \lceil \frac{k}{2} \rceil + 1$.

Now suppose k is odd and assume that $sk(Q(k)) \neq \lceil \frac{k}{2} \rceil + 1$. Then

$$sk(Q(k)) = \left\lceil \frac{k}{2} \right\rceil = \left\lceil \frac{k-1}{2} \right\rceil + 1 = sk(Q(k-1)).$$

By Lemma 2, $R(Q(k))$ contains only non-adjacent rim edges of $Q(k)$. We shall show that $Q(k) - R(Q(k))$ contains a subdivision of $K_{3,3}$ as subgraph, thereby establishing a contradiction (because $Q(k) - R(Q(k))$ is planar).

Note that each internal vertex x_i of $Q(k)$ has three neighbors given by $N(x_i) = \{i, i+k, i+2k\}$ and that each neighbor $j \in N(x_i)$ is incident with two rim edges $(j-1, j)$ and $(j, j+1)$. Here, the operations are reduced modulo $3k$.

For each internal vertex x_i of $Q(k)$, let $N_0(x_i) := \{(j-1,j), (j,j+1) : j \in N(x_i)\}$. That is, $N_0(x_i)$ is the set of rim edges of $Q(k)$ that are incident to the neighbors of x_i.

Suppose there is an internal vertex x_i such that $N_0(x_i) \cap R(Q(k)) = \emptyset$. Let H denote the subgraph of $Q(k) - R(Q(k))$ induced by the set of vertices in $N[x_{i-1}] \cup N[x_i] \cup N[x_{i+1}]$. Here the operations on the subscripts are reduced modulo k. Then it is routine to verify that H is a subdivision of $K_{3,3}$. But this is a contradiction beacuse H is a subgraph of $Q(k) - R(Q(k))$ which is planar.

It remains to consider the case where $N_0(x_i) \cap R(Q(k)) \neq \emptyset$ for any $i = 0, 1, \ldots, k-1$. In this case, since $|N_0(x_i) \cap R(Q(k))| \geq 1$ for any $i = 0, 1, \ldots, k-1$ and $|R(Q(k))| = \lceil \frac{k}{2} \rceil$, it follows (from the pigeonhole principle) that there is precisely only one $0 \leq i \leq k-1$ for which $|N_0(x_i) \cap R(Q(k))| = 2$ and $|N_0(x_j) \cap R(Q(k))| = 1$ for all $j \neq i$. Without loss of generality, let $i = 0$ so that

$$|N_0(x_j) \cap R(Q(k))| = \begin{cases} 2 & \text{if } j = 0 \\ 1 & \text{otherwise} \end{cases} \tag{2}$$

Suppose $(0,1) \in R(Q(k))$.

Then we have $(j, j+1) \notin R(Q(k))$ for any $j \in \{k, -k\}$ because otherwise $|N_0(x_1) \cap R(Q(k))| \geq 2$ and this contradicts Equation (2).

For each internal vertex x_i of $Q(k)$, let $N_1(x_i) := \{(j, j+1) : j \in N(x_i)\}$. With this notation, it follows from Equation (2) that

$$|N_1(x_0) \cap R(Q(k))| = 1 = |N_1(x_{-1}) \cap R(Q(k))| \, .$$

This in turn implies that

$$|N_1(x_1) \cap R(Q(k))| = 0 = |N_1(x_{-2}) \cap R(Q(k))| \, .$$

In fact, for any $i = 0, 1, \ldots, k-1$, we have

$$|N_1(x_i) \cap R(Q(k))| = \begin{cases} 1 & \text{if } i \text{ is even} \\ \\ 0 & \text{if } i \text{ is odd} \end{cases} \tag{3}$$

Let H_0 denote the subgraph of $Q(k)$ induced by the set of vertices in $M = N[x_0] \cup N[x_1] \cup N[x_2]$. Then it is routine to verify that H_0 is a subdivision of $K_{3,3}$.

Suppose $(j, j+1) \in R(Q(k))$ for some $j \in N(x_2)$. Then either $j = 2$ or $j \neq 2$.

Case (1) $j \neq 2$.
Let H_1 denote the subgraph of $Q(k) - R(Q(k))$ induced by the set of vertices in M. Then clearly, H_1 is a subgraph of H_0 with the edge $(0,1)$ deleted. To see that $Q(k) - R(Q(k))$ contains a subdivision of $K_{3,3}$, it suffices to show that there is a path P joining the vertices 0 and 2 and that $E(P) \cap E(H_1) = \emptyset$.

In view of Equation (3), we see that for any vertex x_{2r}, where $2 \leq 2r \leq k-1$, there is a vertex $a_{2r} \in N(x_{2r})$ such that the edges of the path $a_{2r}, a_{2r}+1, a_{2r}+2$ are not in $R(Q(k))$. Denote such path by P_{2r} with P_2 the path with vertices $2, 3, 4$.

Then the path $P^* = P_2 x_4 P_4 \cdots x_{k-3} P_{k-3}$ is a path joining the vertex 2 and the vertex $a \in N(x_{k-1})$ where $a = a_{k-3} + 2$. Note that none of the edges of P^* are in $R(Q(k))$. Let

$$P = \begin{cases} P^* 0 & \text{if } a = 3k-1 \\ \\ P^* x_{k-1} (3k-1) 0 & \text{if } a \neq 3k-1 \end{cases}$$

Then P is a path joining the vertices 2 and 0 having no edges in $R(Q(k))$.

Case (2) $j = 2$.
In this case, we assert that $(2r, 2r+1) \in R(Q(k))$ for all $r = 1, 2, \ldots, \frac{k-3}{2}$.

To see this, suppose there is a smallest even integer s such that $2 \leq s \leq k-5$ and $(s, s+1) \in R(Q(k))$ but $(s+2, s+3) \notin R(Q(k))$. Then $s \in N(x_s)$. Let H_2 be the subgraph of $Q(k) - R(Q(k))$ induced by the set of vertices in $N[x_s] \cup N[x_{s+1}] \cup N[x_{s+2}]$. Let P be a path joining the vertices $s+2$ and s such that P has neither edges in H_2 nor edges in $R(Q(k))$. The existence of such path P can be verified using similar argument as in Case (1). Now, it is routine to check that H_2 together with P is a subdivision of $K_{3,3}$. Hence the assertion follows.

Let H_3 be the subgraph of $Q(k) - R(Q(k))$ induced by the set of vertices in $N[x_{-1}] \cup N[x_0] \cup N[x_1]$. Now, by Lemma 2 and Equations (2) and (3), either (i) $(k-1, k) \in R(Q(k))$ or else (ii) $(2k-1, 2k) \in R(Q(k))$.

(i) Suppose $(k-1, k) \in R(Q(k))$. Then let P_α denote the path whose vertices are $2k+1, 2k+2, \ldots, 3k-2, x_{k-2}, k-2, k-1$ and let P_β denote the path whose vertices are $1, 2, x_2, k+2, k+3, \ldots, 2k-1$. Then $R(Q(k)), H_3, P_\alpha$ and P_β are mutually edge-disjoint. Therefore $H_3 \cup P_\alpha \cup P_\beta$ is a subgraph of $Q(k) - R(Q(k))$ and it forms a subdivision of $K_{3,3}$.

(ii) Suppose $(2k-1, 2k) \in R(Q(k))$. Then let P_α denote the path whose vertices are $k+1, k+2, \ldots, 2k-1$ and let P_β denote the path whose vertices are $1, 2, x_2, 2k+2, 2k+3, \ldots, 3k-2, x_{k-2}, k-2, k-1$. Then $R(Q(k)), H_3, P_\alpha$ and P_β are mutually edge-disjoint. Therefore $H_3 \cup P_\alpha \cup P_\beta$ is a subgraph of $Q(k) - R(Q(k))$ and it forms a subdivision of $K_{3,3}$.

This completes the proof of the lemma. □

Proof of Theorem 1: As was remarked earlier, we just need to show that $sk(Q(k)) = \lceil \frac{k}{2} \rceil + 1$ where $k \geq 4$.

By Corollary 1 and Lemma 4, we have $sk(Q(4)) = 3$. The result then follows from Lemma 5 by induction on k. □

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Some Conditions for the Existence of (d, k)-Digraphs

Yus Mochamad Cholily[⋆], Edy Tri Baskoro, and Saladin Uttunggadewa

Department of Mathematics, Institut Teknologi Bandung (ITB),
Jalan Ganesa 10 Bandung 40132, Indonesia
{yus, ebaskoro, s_uttunggadewa}@dns.math.itb.ac.id

Abstract. A (d, k)-*digraph* is a diregular digraph of degree $d \geq 4$, diameter $k \geq 3$ and the number of vertices $d+d^2+\cdots+d^k$. The existence problem of (d, k)-digraphs is one of difficult problem. In this paper, we will present some new necessary conditions for the existence of such digraphs.

1 Introduction

Let G be a digraph. We write $V(G)$ or V for the vertex set of G and $A(G)$ or A for the edge set of G. The *distance* $\delta(u, v)$ from vertex u to vertex v is defined as the length of a shortest walk from u to v. The *diameter* k of digraph G is the maximum distance between any two vertices in G.

A set of all vertices at distance i from vertex v is denoted by $N^i(v)$ and a set of all vertices at distance i to vertex v is denoted by $N^{-i}(v)$. In particular, for $i = 1$ and $i = -1$ we use $N^+(v)$ for $N^1(v)$ and $N^-(v)$ for $N^{-1}(v)$. If $|N^+(v)| = |N^-(v)| = d$ for each $v \in V(G)$ then the digraph G is called *diregular* of degree d.

In the design of interconnection networks it is interesting to find digraphs with the largest number of vertices n given values of maximum out-degree d and diameter k (see [6],[4],[7]). It is easy to see that

$$n \leq M_{d,k} = 1 + d + d^2 + \cdots + d^k.$$

This (trivial) upper bound is called the *Moore bound*. It is well known that this bound is attained *only* for $d = 1$ by the cycle digraph of order $k + 1$, and for $k = 1$ by the complete digraph K_{d+1} (see [5], [13]). This motivated the study of the existence of digraphs with order 'close' to the Moore bound, for $d \geq 2$, $k \geq 2$.

A (d, k)-*digraph* is a diregular digraph of degree $d \geq 2$, diameter $k \geq 2$ and the number of vertices n one less than the *Moore* bound, that is, $n = d+d^2+...+d^k$. Since the order is one less than the Moore bound, every (d, k)-digraph G has the characteristic property that for every vertex $x \in G$ there exists exactly one

⋆ Permanent address : Department of Mathematics, Muhammadiyah University of Malang. Jl. Tlogomas 246 Malang 65144 Indonesia. Email : yus@umm.ac.id

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vertex y such that there are two walks of length $\leq k$ from x to y. Such a vertex y is called the *repeat* of x, denoted by $r(x) = y$. In case $r(u) = u$, vertex u is called a *selfrepeat* (the two walks, in this case, have length 0 and k). Moreover, Baskoro, Miller and Plesník [2] showed that the function r is an automorphism on $V(G)$. For any integer $p \geq 1$, define $r^p(v) = r(r^{p-1}(v))$ with $r^0(v) = v$. Then, for every vertex v of G, there exists a smallest natural number $\omega(v)$ called the *order* of v, such that $r^{\omega(v)}(v) = v$.

The study of the existence of (d, k)-digraphs has received much attention. For $k = 2$, Fiol, Allegre and Yebra [6] showed the existence of $(d, 2)$-digraphs for $d \geq 2$. In particular, for $d = 2$ and $k = 2$, Miller and Fris [10] proved that there are exactly three non-isomorphic $(2, 2)$-digraphs. Further, Gimbert [9] showed that there is only one $(d, 2)$-digraph, namely the line digraph $L(K_{d+1})$ of the complete digraph K_{d+1}, for $d \geq 3$. For diameter $k \geq 3$, it is known that there are no $(2, k)$-digraphs [11]. Recently, it was proved that there are no $(3, k)$-digraphs with $k \geq 3$ [3]. Thus, the remaining case still open is the existence of (d, k)-digraphs with $d \geq 4$ and $k \geq 3$.

Several necessary conditions for the existence of (d, k)-digraphs have been obtained (see [1],[2],[8]). One such condition is that any (d, k)-digraph contains either exactly k selfrepeats or none, $k \geq 3$ [2]. In this paper we present some new necessary conditions for the existence of (d, k)-digraphs as stated in the following theorems.

Theorem 1. *Let G be a (d, k)-digraph containing selfrepeats, $d \geq 4$, $k \geq 3$. Let v be a selfrepeat of G. If $N^+(v)$ contains a vertex of order $d - 1$ then G consists of only vertices of orders either 1 or $d - 1$. More precisely, G consists of exactly k selfrepeats and $M_{d,k} - 1 - k$ vertices of order $d - 1$.*

In particular, for the case of $d = 4$ we have the following corollary.

Corollary 1. *If a $(4, k)$-digraph G exists, $k \geq 3$, then either G contains no selfrepeats or G contains exactly k selfrepeats and $M_{d,k} - 1 - k$ vertices of order 3.*

Theorem 2. *Let G be a (d, k)-digraph containing selfrepeats, $d \geq 4$, $k \geq 3$. Let v be a selfrepeat of G. Then the existence of any two vertices x, y in $N^+(v)$ of orders m and n implies the existence of a vertex of order* $\mathrm{lcm}(m, n)$*, the least common multiple of m and n, in G.*

Furthermore, if the degree $d = 1 + m + n$ and $N^+(v)$ contains only vertices of orders either $1, m$ and n then the following theorem holds.

Theorem 3. *Let G be a (d, k)-digraph containing selfrepeats, $d \geq 4$, $k \geq 3$. Let v be a selfrepeat of G. If $N^+(v)$ contains only vertices of orders 1, m and n (and $d = 1 + m + n$) then G consists of only vertices of orders 1, m, n and* $\mathrm{lcm}(m, n)$.

For some values of k, we show the non-existence of $(5, k)$-digraphs.

Theorem 4. *For $k \in \{3, 7, 9, 11, 12, 14, 17, 19, 21, 22\}$ there are no $(5, k)$-digraphs containing selfrepeats.*

2 Auxiliary Results

In this section, we shall give insights into the structure of a (d, k)-digraph.

Lemma 1. *Let $(v_0, v_1, \cdots, v_p)$ be a walk W of length $p < k$ in a (d, k)-digraph G. If $\omega(v_0) = m$ and $\omega(v_p) = n$ (m and n not necessarily distinct) then for each vertex $x \in W$, $\omega(x)$ must divide* $\mathrm{lcm}(m, n)$.

Proof. Let $t = \mathrm{lcm}(m, n)$. For a contradiction, let there be $j \geq 1$ such that $\omega(v_j)$ does not divide t. Then, $r^t(v_j) \neq v_j$. Thus we have two distinct walks of length $< k$ from v_0 to v_j, namely $(v_0, \cdots, v_j, \cdots, v_p)$ and $(v_0 = r^t(v_0), ..., r^t(v_j), ..., v_p = r^t(v_p))$. This is a contradiction since there exists at most one walk of length $< k$ between any two vertices in a (d, k)-digraph. □

Lemma 2. *Let $(v_0, v_1, \cdots, v_p)$ be a walk W of length $p \leq k$ in a (d, k)-digraph G. If $\omega(v_0) = m$ and $\omega(v_p) = n$ (m and n not necessarily distinct) and $r(v_0) \neq v_p$ then for each vertex $x \in W$, $\omega(x)$ must divide* $\mathrm{lcm}(m, n)$.

Proof. Let $t = \mathrm{lcm}(m, n)$. Assume that there exists $j \geq 1$ such that $\omega(v_j)$ does not divide t. Then, $r^t(v_j) \neq v_j$. Thus, we have two distinct walks of length $\leq k$ from v_0 to v_j, namely $(v_0, \cdots, v_j, \cdots, v_p)$ and $(v_0 = r^t(v_0), \cdots, r^t(v_j), \cdots, v_p = r^t(v_p))$. This is impossible since $r(v_0) \neq v_p$. □

In the special case of v_0 being a selfrepeat ($m = 1$), we have the following consequence. Let $(v_0, v_1, ..., v_p)$ be a walk of length $p \leq k$ in a (d, k)-digraph G. By applying Lemma 2 on the subwalk (v_0, v_1, v_2) we get $\omega(v_1)|\omega(v_2)$. Next, applying Lemma 2 on the subwalk (v_0, v_1, v_2, v_3), we obtain $\omega(v_2)|\mathrm{lcm}(\omega(v_0), \omega(v_3))$. This means that $\omega(v_2)|\omega(v_3)$. Again, by applying Lemma 2 repeatedly on the walks $(v_0, v_1, v_2, \cdots, v_j)$, for $j = 4, 5, \cdots, p$, we get a sequence of orders satisfying $\omega(v_0)|\omega(v_1)$, $\omega(v_1)|\omega(v_2)$, $\cdots$, $\omega(v_{p-1})|\omega(v_p)$. A similar result can be obtained if v_p is a selfrepeat. Note that if both v_0 and v_p are selfrepeats then it is easy to see that $\omega(v_i) = 1$ for each i (by applying Lemma 2 and Lemma 1 (in that order). Thus, the following corollary holds.

Corollary 2. (Lemma 2 in [2]) *Let $(v_0, v_1, ..., v_p)$ be a walk of length $p \leq k$ in a (d, k)-digraph G. If v_0 or v_p is a selfrepeat then the sequence of orders $\omega(v_0), \omega(v_1), ..., \omega(v_p)$ is monotonically divisible, that is, $\omega(v_0)|\omega(v_1)$, $\omega(v_1)|\omega(v_2), \cdots$, $\omega(v_{p-1})|\omega(v_p)$, if v_0 is a selfrepeat, or $\omega(v_p)|\omega(v_{p-1})$, $\omega(v_{p-1})|\omega(v_{p-2})$, $\cdots$, $\omega(v_1)|\omega(v_0)$, if v_p is a selfrepeat.* □

The following corollary appeared as Corollary 5 in [2]. In this paper we give an alternative and shorter proof below.

Corollary 3. *For any selfrepeat v in a (d, k)-digraph, the permutation r of $N^+(v)$ has the same cycle structure as the permutation r of $N^-(v)$.*

Proof. Since v is a selfrepeat, there exist d pairwise internally disjoint walks $(v, z_1, \cdots, y_1, v)$, $(v, z_2, \cdots, y_2, v)$, $\cdots$, $(v, z_d, \cdots, y_d, v)$ of lengths $\leq k + 1$. By applying Lemma 2 on $(v, z_1, \cdots, y_1)$, we get $\omega(z_1)|\omega(y_1)$. Applying Lemma 2 to $(z_1, \cdots, y_1, v)$ we obtain $\omega(y_1)|\omega(z_1)$. Thus $\omega(z_1) = \omega(y_1)$. Similarly, we will get $\omega(z_i) = \omega(y_i)$ for any $i = 2, 3, \cdots, d$. □

We distinguish vertices in a (d,k)-digraph, depending on the distance between a vertex and its repeat as follows. Vertex x is said to be 0-*type vertex* if $\delta(x,r(x)) < k$ and otherwise ($\delta(x,r(x)) = k$), vertex x is called a k-*type vertex.*

Lemma 3. *Let v be a 0-type vertex of a (d,k)-digraph G where $\omega(v) = p$. If W_1 and W_2 are the two walks of lengths $\leq k$ from v to $r(v)$ then the order of any internal vertex of each walk must divide p.*

Proof. Since v is 0-type vertex then either $p = 1$ (v is a selfrepeat) or $p > 1$. If $p = 1$, the two walks are $W_1 = (v)$ and W_2 is part of a directed cycle C_k containing selfrepeat vertices only. Thus, the assertion follows. Now, let $p > 1$. Since v is 0-type vertex then the two walks from v to $r(v)$ will have different lengths. One, say W_1, is of length $< k$ and the other, say W_2, is of length k. By Lemma 1, the orders of all internal vertices of W_1 must divide p. Next, we shall show that the orders of all internal vertices of W_2 also divide p. For a contradiction, let u be a vertex in W_2 where $\omega(u)$ does not divide p. Therefore, $r^p(u) \neq u$ which implies that $r(u) \notin W_1$. Thus, there are three different walks W_1, W_2 and $(v, ..., r^p(u), ..., r(v))$ of lengths $\leq k$ from v to $r(v)$, a contradiction. □

Lemma 4. *Let v be a k-type vertex of a (d,k)-digraph G where $\omega(v) = p$. If $W_1 = (v, u_1, u_2, \cdots, u_{k-1}, r(v))$ and $W_2 = (v, w_1, w_2, \cdots, w_{k-1}, r(v))$ are the two distinct walks of length k from v to $r(v)$ then, for all $j \in \{1, 2, \cdots, k-1\}$, $\omega(u_j)$ and $\omega(w_j)$ must each divide $2p$.*

Proof. For a contradiction, let there be $j \geq 1$ such that $\omega(u_j)$ does not divide $2p$. Then $r^{2p}(u_j) \neq u_j$. Since $u_j \neq r(v)$, we have $r^{2p}(u_j) \notin W_2$. Hence there exist three different walks W_1, W_2 and $(v, \cdots, r^{2p}(u_j), \cdots, r(v))$ of lengths $\leq k$ connecting v to $r(v)$, a contradiction. □

3 Proofs of the Main Results

Proof of Theorem 1. Let v be a selfrepeat vertex of a (d,k)-digraph G. Let $N^+(v)$ contain a vertex of order $d-1$. Since $r(N^+(v)) = N^+(r(v)) = N^+(v)$ and G is of degree d then $N^+(v)$ consists of exactly one selfrepeat and $d-1$ vertices of order $d-1$. Since the diameter of G is k, then $V(G) = \bigcup_{i=0}^{k} N^i(v)$. Let y be an arbitrary vertex in $N^2(v)$ and (v, x, y) be a walk of length 2 connecting v to y in G. Lemma 2 guarantees that $\omega(x)|\omega(y)$. Since each vertex in $N^+(x)$ must be at distance at most k to v then there are d walks of length $\leq k+1$ from x to v. One of the d walks, say W, will contain y. Let $W = (x, y, \cdots, z, v)$. By Corollary 3, $\omega(z)$ is either 1 or $d-1$. By considering the walk $(y, \cdots, z, v)$, Lemma 2 implies that $\omega(z)|\omega(y)$. Now, consider the walk $(x, y, \cdots, z)$. Again, by Lemma 2, we have $\omega(y)|\mathrm{lcm}(\omega(x), \omega(z))$. As we know that $\omega(x)$ and $\omega(z)$ are either 1 or $d-1$, then $\omega(y)$ is either 1 or $d-1$ too. Note that if $\omega(x) = d-1$ then $\omega(y)$ must be $d-1$ too since otherwise $r(v) = y$, a contradiction with v being a selfrepeat. Since y was chosen arbitrarily, we conclude that any vertex of $N^2(v)$ has order either 1 or $d-1$. A similar process can be applied to show that any vertex of $N^i(v)$, for each $i = 3, 4, ..., k$, has order either 1 or $d-1$. □

Proof of Corollary 1. Let G be a $(4, k)$-digraph, $k \geq 3$. If G contains a selfrepeat v, then the only possible permutation cycle structure of repeats on $N^+(v)$ is $(1, 3)$. By Theorem 1, G consists of exactly k selfrepeats and $M_{d,k} - 1 - k$ vertices of order 3. Therefore, the theorem follows. □

Proof of Theorem 2. Let G be a (d, k)-digraph containing selfrepeats. Let v be a selfrepeat of G. By Corollary 3, $N^-(v)$ contains vertices of orders $1, m$ and n too. Let x be a vertex of $N^+(v)$ where $\omega(x) = m$ and z be a vertex of $N^-(v)$ where $\omega(z) = n$. Let $m \neq n$. We shall show that there exists a vertex y such that $\omega(y) = \text{lcm}(m, n)$. Since a vertex v is a selfrepeat and $r(x) \notin N^-(v)$ then there are d internally disjoint walks of length $\leq k+1$ from x to v. So, one of them must contain z. Let $W = (x, y, \cdots, z, v)$ be the walk from x to v containing z. Applying Lemma 2 on walk $(x, y, \cdots, z)$, we get $\omega(y)|\text{lcm}(\omega(x), \omega(z))$, that is $\omega(y)|\text{lcm}(m, n)$. Again, using Lemma 2 on $(y, \cdots, z, v)$, we have $\omega(z)|\text{lcm}(\omega(y), \omega(v))$ (which implies $n|\omega(y)$). Thus, $\omega(y)$ is either n or $\text{lcm}(m, n)$. However, $\omega(y) \neq n$ since otherwise there are two different walks (v, x, y) and $(r^n(v) = v, r^n(x), r^n(y) = y)$ of length $\leq k$ (which implies that $r(v) = y$), a contradiction. Hence, there exists $y \in G$ such that $\omega(y) = \text{lcm}(m, n.)$ □

The next proof will show that if v is a selfrepeat in a (d, k)-digraph G and the permutation cycle structures of r on $N^+(v)$ is $(1)(m)(n)$ with $d = 1 + m + n$ then G consists of only vertices of orders $1, m, n$ and $\text{lcm}(m, n)$.

Proof of Theorem 3. By Theorem 2, the existence of a vertex of order $\text{lcm}(m, n)$ is guaranteed. Next, we shall show that G consists of only vertices of orders 1,m,n and $\text{lcm}(m, n)$. Since the diameter of G is k then $V(G) = \bigcup_{i=0}^{k} N^i(v)$. To prove the theorem, we have to show that any vertex in $N^i(v)$, $i = 1, 2, \cdots, k$, has order either 1,m,n or $\text{lcm}(m, n)$. For $i = 1$, it holds by the premise of the theorem. Now, let $y \in N^2(v)$ and (v, x, y) be the walk of length 2 in G. By Lemma 1, $\omega(x)|\omega(y)$. Furthermore, there exists a walk W of length less than or equal to $k+1$ from x to v containing y. Let $W = (x, y, \cdots, z, v)$. By Corollary 3, $\omega(z)$ is either 1,m or n. So, $\omega(z)|\omega(y)$ by applying Lemma 2 on walk $(y, \cdots, z, v)$. Since $r(x) \neq z$, then by applying Lemma 2 on walk $(x, y, \cdots, z)$, we get $\omega(y)|\text{lcm}(\omega(x), \omega(z))$. Thus $\omega(y)$ is either $1, m$,n or $\text{lcm}(m, n)$. Since y was chosen arbitrarily, then $\omega(y) = 1, m, n$ or $\text{lcm}(m, n)$ holds for any $y \in N^2(v)$. A similar process can be applied to show that $\omega(y)$ is either $1, m, n$, or $\text{lcm}(m, n)$ for any $y \in N^i(v)$, $i = 3, ..., k$. □

Gimbert ([8], Proposition 1) showed that the number of permutation cycles with even length in a (d, k)-digraph must be a multiple of k. By this fact, the next proof will show that there are no $(5, k)$-digraphs for some values of k.

Proof of Theorem 4. Let G be a $(5, k)$-digraph containing a selfrepeat, with $k \in \{3, 7, 9, 11, 12, 14, 17, 19, 21, 22\}$. Since the diameter of G is k, then G must contains exactly k selfrepeats (by Theorem 4, [2]). Let u be a selfrepeat of G. Then $N^+(u)$ contains exactly one selfrepeat. Since r is an automorphism, $r(N^+(u)) = N^+(u)$, then the only possible permutation cycle structures of repeats on $N^+(u)$ are either (1)(4) or (1)(2)(2).

If the permutation cycle structures of r on $N^+(u)$ is (1)(4) then, by Theorem 1, the order of all vertices other than selfrepeats in G is 4. So, the number of permutation cycles of length 4 is $\frac{M_{d,k}-1-k}{4}$. In the second case, Theorem 3 guarantees that all vertices other than selfrepeats have order 2. Therefore, the number of permutation cycles of length 2 is $\frac{M_{d,k}-1-k}{2}$. In Table 1, we summarize the number of permutation cycles of lengths 2 and 4 for those values of k.

Table 1. The number of permutation cycles of lengths 2 and 4

k	$\lvert V(G)\rvert - k$	Number of permutation cycles of length 2	Number of permutation cycles of length 4
3	152	76	38
7	97648	48824	24412
9	2441396	1220698	610349
11	61035144	30517572	15258786
13	1525878892	762939446	381469723
14	7629394516	3814697258	1907348629
17	953674316388	4.76837E+11	2.38419E+11
19	23841857910136	1.19209E+13	5.96046E+12
21	596046447753884	2.98023E+14	1.49012E+14
22	2980232238769510	1.49012E+15	7.45058E+14

In Table 1, in each case the number of permutation cycles not a multiple of k. Therefore, by Proposition 1 from [8], there are no $(5, k)$-digraphs for the above values of k. □

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Subdivision Number of Large Complete Graphs and Large Complete Multipartite Graphs

Severino V. Gervacio

Department of Mathematics, De La Salle University,
2401 Taft Avenue, 1004 Manila, Philippines
gervacios@dlsu.edu.ph

Abstract. A graph whose vertices can be represented by distinct points in the plane such that points representing adjacent vertices are 1 unit apart is called a *unit-distance graph*. Not all graphs are unit distance graphs. However, if every edge of a graph is subdivided by inserting a new vertex, then the resulting graph is a unit-distance graph. The minimum number of new vertices to be inserted in the edges of a graph G to obtain a unit-distance graph is called the *subdivision number of* G, denoted by $\mathrm{sd}\,(G)$. We show here in a different and easier way the known result $\mathrm{sd}\,(K_{m,n}) = (m-1)(n-m)$ when $n \geq m(m-1)$. This result is used to show that the subdivision number of the complete graph is asymptotic to $\binom{n}{2}$, its number of edges. Likewise, the subdivision number of the complete bipartite graph $K_{m,n}$ is asymptotic to mn, its number of edges. More generally, the subdivision number of the complete n-partite graph is asymptotic to its number of edges.

1 Introduction

By a *graph* G we mean an ordered pair $G = \langle V(G), E(G)\rangle$ where $V(G)$ is a finite nonempty set whose elements are called *vertices* and $E(G)$ is a set of 2-element subsets of $V(G)$, called *edges*. An edge consisting of the vertices x and y will be denoted by the symbol $[x, y]$.

One unary operation on graphs that we shall deal with mainly is *edge subdivision.*

Definition 1. *Let $e = [a, b]$ be an edge of a graph G. To* subdivide *e means to introduce a new vertex, say c, and replace the edge e by two new edges $[a, c]$ and $[c, b]$.*

The operation of edge subdivision may also be thought of as inserting a vertex in an edge. This operation may be applied iteratively to a graph G. However, we shall not do this. To subdivide k edges of G means to subdivide k distinct edges of G.

Example 1. Figure 1 illustrate the subdivision of the edge $e = [a, b]$ of the graph G, obtaining the graph G'.

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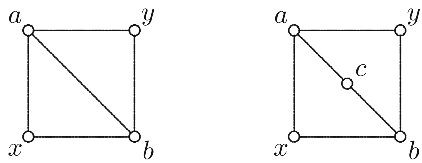

Fig. 1. Subdividing the edge $[a, b]$ of G to obtain the graph G'

$$\phi : \begin{cases} x \to (0, 0) \\ a \to (0, 1) \\ b \to (\frac{\sqrt{3}}{2}, \frac{1}{2}) \\ y \to (\frac{\sqrt{3}}{2}, \frac{3}{2}) \end{cases}$$

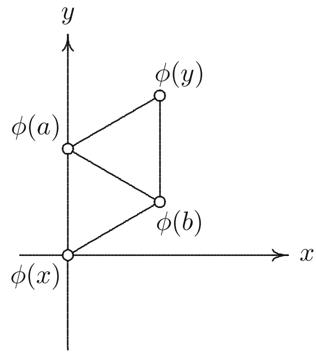

Fig. 2. Unit-distance representation of a graph

Definition 2. *Let G be a graph and R^2 the Euclidean space of dimension 2. A mapping $\phi: V(G) \to R^2$ is called a* unit-distance representation of G *if ϕ is one-to-one and the distance between $\phi(x)$ and $\phi(y)$ is 1 whenever $[x, y] \in E(G)$. The graph G is caled a* unit-distance graph *if there exists a unit-distance representation of G.*

Example 2. The graph G in Figure 1 is a unit-distance graph. One unit-distance representation of G is shown in Figure 2

We next look at graphs that are not unit-distance graphs.

Example 3. It is easy to verify that the complete graph K_4 and the complete bipartite graph $K_{2,3}$ are not unit-distance graphs.

Suppose that K_4 is a unit-distance graph. Any three vertices of K_4 are represented by three points forming the vertices of an equilateral triangle with unit sides. The fourth vertex of K_4 must be represented by a point equidistant from the vertices of this equilateral triangle. But that point is less than 1 unit away from each vertex of the equilateral triangle. Therefore, K_4 does not have any unit-distance representation. Hence it is not a unit-distance graph. A similar argument can be used to prove that $K_{2,3}$ is not a unit-distance graph.

It follows that all complete graphs K_n with $n \geq 4$ are not unit-distance graphs. Likewise, every complete bipartite graph $K_{m,n}$ with $m \geq 2$ and $n \geq 3$ are not unit-distance graphs.

Lemma 1. *Let G be any graph with $m > 0$ edges. The graph G' obtained from G by subdividing all its edges is a unit-distance graph.*

Proof. Let G be a graph of order n and consider K_n. It is enough to show that by subdividing all the edges of K_n, we obtain a unit-distance graph. Let the vertices of K_n be $x_1, x_2, \ldots, x_n$. Let x_{ij} be the subdivision vertex of the edge $[x_i, x_j]$. Let the vertices $x_1, x_2, \ldots, x_n$ be mapped to the n points $p_i = (\frac{i}{n}, 0)$, $i = 1, 2, \ldots, n$ respectively. Note that the distance between any two points p_i and p_j is $|i-j|/n < 1$. For each pair of distinct points p_i and p_j, we can determine a point p_{ij} that is 1 unit distance away from both p_i and p_j. Let p_{ij} be the point assigned to the subdivision vertex x_{ij}. Then the mapping $\phi(x_i) = p_i$, $\phi(x_{ij}) = p_{ij}$ is a unit-distance representation of the subdivision of K_n. Since G is isomorphic to a subgraph of K_n, it follows that G' is a unit-distance graph.

Lemma 1 guarantees that given any graph G, there exists a minimum number of edges of G to be subdivided to obtain a unit-distance graph. Let G be a graph. The *subdivision number of* G, denoted by $\mathrm{sd}\,(G)$, is

Definition 3. *the minimum number of edges of* G *to be subdivided to obtain a unit-distance graph.*

Note that $\mathrm{sd}\,(G) = 0$ if and only if G is a unit-distance graph. We shall determine asymptotic formulas for $\mathrm{sd}\,(K_n)$ and $\mathrm{sd}\,(K_{m,n})$.

2 Preliminary Results

We shall begin by proving a lemma that will be used to obtain asymptotic formulas for the subdivision number of K_n as well as $K_{m,n}$.

Lemma 2. *If $n \geq m(m-1)$, then there exists a unit-distance bipartite graph G, which is a spanning subgraph of $K_{m,n}$, that has $n + m(m-1)$ edges. Furthermore, if $A \cup B$ is a bipartition of the vertices in G, where $|A| = m$, $|B| = n$ then there exists a unit-distance representation of G such that each point in the representation corresponding to a vertex in B is less than 2 unit-distances away from each point representing a vertex in A.*

Proof. We shall construct a unit-distance representation of the required bipartite graph. Consider m points $p_i = (\frac{i}{n}, 0)$, $1 \leq i \leq m$. For each pair p_i, p_j of distinct points, we can find exactly two new points in the plane, each 1 unit away from both p_i and p_j. This is so because the distance between p_i and p_j is less than 1. Please refer to Figure 3. Let these two points be denoted by p_{ij} and q_{ij}. Since there are exactly $\binom{m}{2}$ pairs of distinct points, the total number of new points that we obtain is $t = 2\binom{m}{2} = m(m-1)$. We now have a unit-distance bipartite graph that is a spanning subgraph of $K_{t,m}$. Furthermore, this bipartite graph has $2t$ edges. Now, $n \geq m(m-1) = t$. Add $n - t$ more new points such that each of them is 1 unit-distance away from p_1 and on the right side of the vertical line through p_1. We now have a unit-distance bipartite graph that is a spanning subgraph of $K_{m,n}$. The total number of edges of our bipartite graph is $2t + n - t = n + t = n + m(m-1)$. Note that by our construction, each point p_{ij} or q_{ij} is less than 2 unit-distances away from any of the points $p_1, p_2, \ldots, p_n$.

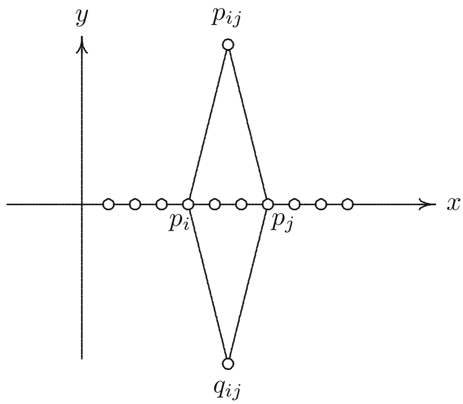

Fig. 3. Constructing a unit-distance spanning subgraph of $K_{m,n}$

More precisely, the distance between any point we have determined different from $p_1, p_2, \ldots, p_m$ to any of the points $p_1, p_2, \ldots, p_m$ is less than $\sqrt{2}$.

The unit-distance graph constructed in Lemma 2 is one with maximum number of edges as we will show in the next lemma.

Lemma 3. *Let $n \geq m(m-1)$. Then every spanning subgraph of $K_{m,n}$ which is a unit-distance graph has at most $n + m(m-1)$ edges.*

Proof. Let G be a spanning subgraph of $K_{m,n}$ which is a unit-distance graph. Let $A \cup B$ be a bipartition of the vertices of G into independent subsets A and B with m and n vertices respectively. Without loss of generality, we may assume that each vertex in B has positive degree. Let us consider the following cases:

Case 1. For each $b \in B$, $\deg(b) \leq 2$. Let t be the number of vertices with degree 2. Then, at most $2t$ edges are contributed by these vertices. The remaining vertices have degree at most 1. Therefore, since there are $n-t$ of them, the number of edges they contribute is at most $n-t$. Thus, G has at most $2t+n-t = n+t$ edges. But the maximum possible number of vertices with degree 2 is $2\binom{m}{2} = m(m-1)$. Hence G has at most $n + t \leq n + m(m-1)$ edges.
Case 2. There exits $b \in B$ with $\deg(b) = k \geq 3$. Let $a_1, a_2, \ldots, a_k$ be the neighbors of b. Please refer to Figure 4. Since G is a unit-distance graph, then for each pair a_i, a_j of vertices in A, at most two vertices in B can be adjacent to both a_i and a_j.

Remove the edge $[b, a_1]$. Add two new vertices x and y to the set B. Let x be adjacent to a_1 and a_3, and let y be adjacent to a_1 and a_2. It is still true that for each pair of vertices in A, at most two vertices in B are adjacent to both vertices in the pair. We have added four new edges and deleted one existing edge. Delete two vertices in B with degree 1. We shall show later that we have these vertices to delete. Note that this modification increases the number of edges by 1. The set A is unchanged while the number of vertices in B is preserved. It is not important at this point to know whether the graph obtained in this process is a unit-distance graph or not. We shall ask this question after the construction

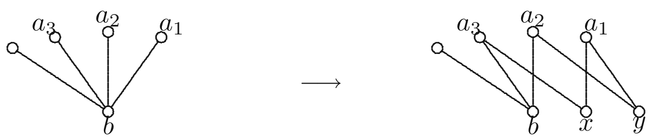

Fig. 4. Modifying the graph at a vertex of degree greater than 2

is finished. We repeat this process until no vertex in B has degree greater than 2. The total number of vertices in B with degree 2 does not exceed 2 times the total number of pairs of vertices in A, which is equal to $m(m-1)$. But the total number of vertices in B is $n \geq m(m-1)$. Thus, each time we modify the graph by introducing two new vertices, we can find two vertices of degree 1 to delete. The resulting graph now has a new set B' with n vertices having degrees 1 or 2. We can therefore associate m vertices in A with the p_i's as in Lemma 2. The n vertices in B' can be associated with n points as in Lemma 2 also. If k is the number of vertices in B' with degree 2, then $n-k$ is the number of vertices in B' with degree 1. The number of edges in the modified graph is then equal to $2k + (n-k) = n + k \leq n + 2\binom{m}{2} = n + m(m-1)$.

We now state and prove an important result. This result can be found in [1] with a slightly different proof.

Theorem 1. *Let* $n \geq m(m-1)$. *Then* $\operatorname{sd}(K_{m,n}) = (m-1)(n-m)$.

Proof. By Lemma 2, we can construct a spanning subgraph G of $K_{m,n}$ which is a unit-distance graph having $n + m(m-1)$ edges. Let $A \cup B$ be a bipartition of the vertex set of G into independent sets where $|A| = m$ and $|B| = n$. By Lemma 3, this subgraph has the maximum number of edges among all spanning unit-distance subgraphs. Furthermore, by Lemma 2, we can find a unit-distance representation of G such that every point representing a vertex in B is less than 2 unit-distances away from each point representing a vertex in A. Thus, if an edge $[a,b]$ is not yet in G, we can locate a point in the plane that is 1 unit-distance from each of the points representing a and b. It follows that $\operatorname{sd}(K_{m,n})$ is equal to the number of edges in $K_{m,n}$ that are not in G. Hence, $\operatorname{sd}(K_{m,n}) = mn - [n + m(m-1)] = (m-1)(n-m)$.

3 Main Results

Formulas giving the exact values of $\operatorname{sd}(G)$, where G is the complete graph or the complete bipartite graph are not known. Here, we shall give asymptotic formulas for the subdivision number of the complete graph, complete bipartite graph and complete n-partite graph.

Theorem 2.

$$\lim_{n\to\infty} \frac{\mathrm{sd}\,(K_{n,n})}{n^2} = 1.$$

Proof. Assume n is sufficiently large and let $m = \lfloor\sqrt{n}\rfloor$. Then $n \geq m(m-1)$. To see this, suppose that $n < m(m-1)$. Then $n < \sqrt{n}(\sqrt{n}-1) = n - \sqrt{n}$, which is a contradiction.

Now let A, B form a partition of the vertex set of $K_{n,n}$ into two independent n-sets. Take $u = \left\lfloor \dfrac{n}{m} \right\rfloor$ disjoint subsets of B, each having m elements. Let these u subsets be denoted by $B_1, B_2, \ldots, B_u$. Consider the subgraph of $K_{n,n}$ induced by $A \cup B_i$. This is the complete bipartite graph $K_{m,n}$ whose subdivision number is $(m-1)(n-m)$. The number of these subgraphs of $K_{m,n}$ is equal to u. Thus, we have to subdivide at least $u(m-1)(n-m)$ edges of $K_{n,n}$ to obtain a unit-distance graph. Therefore, we have

$$\frac{\mathrm{sd}\,(K_{n,n})}{n^2} \geq \frac{u(m-1)(n-m)}{n^2}$$

But $u = \left\lfloor \dfrac{n}{m} \right\rfloor \geq \left\lfloor \dfrac{n}{\sqrt{n}} \right\rfloor = \lfloor\sqrt{n}\rfloor$. Therefore,

$$\begin{aligned}\frac{\mathrm{sd}\,(K_{n,n})}{n^2} &\geq \frac{\lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor - 1)(n - \lfloor\sqrt{n}\rfloor)}{n^2} \\ &= \frac{\lfloor\sqrt{n}\rfloor}{\sqrt{n}}\,\frac{\lfloor\sqrt{n} - 1\rfloor}{\sqrt{n}}\,\frac{n - \lfloor\sqrt{n}\rfloor}{n}\end{aligned}$$

It follows that $\displaystyle\lim_{n\to\infty} \frac{\mathrm{sd}\,(K_{n,n})}{n^2} \geq 1$ since each quotient in our last expression has limit 1 as $n \to \infty$. Since $\mathrm{sd}\,(K_{n,n}) \leq n^2$, the theorem follows.

We shall use essentially the same argument to prove our next result.

Theorem 3.

$$\lim_{m,n\to\infty} \frac{\mathrm{sd}\,(K_{m,n})}{mn} = 1.$$

Proof. Since $K_{m,n}$ is isomorphic to $K_{n,m}$, we may assume without loss of generality that $n \geq m$. Let $\hat{m} = \lfloor\sqrt{m}\rfloor$. It is easy to check that $n \geq \hat{m}(\hat{m}-1)$. Let A, B be the two independent sets forming the vertex set of $K_{m,n}$, with $|A| = n$ and $|B| = m$. Take $u = \left\lfloor \frac{m}{\hat{m}} \right\rfloor$ disjoint subsets of B having $\hat{m}$ vertices each. Label these subsets $B_1, B_2, \ldots, B_u$. Each pair A, B_i forms a subgraph $K_{\hat{m},n}$ of $K_{m,n}$ and these u subgraphs are edge-disjoint. Furthermore $\mathrm{sd}\,(K_{\hat{m},n}) = (\hat{m}-1)(n-\hat{m})$. It follows that $\mathrm{sd}\,(K_{m,n}) \geq u(\hat{m}-1)(n-\hat{m})$. Since $u \geq \lfloor\sqrt{m}\rfloor$, we have

$$\begin{aligned}\frac{\mathrm{sd}\,(K_{m,n})}{mn} &\geq \frac{\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor - 1)(n - \lfloor\sqrt{m}\rfloor)}{mn} \\ &= \frac{\lfloor\sqrt{m}\rfloor}{\sqrt{m}}\,\frac{\lfloor\sqrt{m}\rfloor - 1}{\sqrt{m}}\,\frac{n - \lfloor\sqrt{m}\rfloor}{n}\end{aligned}$$

Each of the three fractions in the lower bound above has limit 1 as $m, n \to \infty$. Therefore, $\lim_{m,n\to\infty} \dfrac{\mathrm{sd}\,(K_{m,n})}{mn} \geq 1$. Since the $\mathrm{sd}\, K_{m,n} \leq mn$, the theorem now follows.

Theorem 4. *For each fixed integer $m > 1$,*

$$\lim_{n\to\infty} \frac{\mathrm{sd}\,(K_{m,n})}{mn} = 1 - \frac{1}{m}.$$

Proof. Since $n \to \infty$, we may assume that $n \geq m(m-1)$. Therefore, $\mathrm{sd}\,(K_{m,n}) = (m-1)(n-m)$ and so

$$\lim_{n\to\infty} \frac{\mathrm{sd}\,(K_{m,n})}{mn} = \lim_{n\to\infty} \frac{(m-1)(n-m)}{mn} = 1 - \frac{1}{m}.$$

We shall apply Theorem 2 to prove the next result.

Theorem 5.

$$\lim_{n\to\infty} \frac{\mathrm{sd}\,(K_n)}{\binom{n}{2}} = 1.$$

Proof. Consider the complete graph K_n. Let $m = \lfloor \sqrt{n} \rfloor$. Take m disjoint subsets $A_1, A_2, \ldots, A_m$ of $V(K_n)$ with each $|A_i| = m$. For each pair A_i, A_j of distinct subsets, we form the complete bipartite graph $K_{m,m}$. This is a subgraph of K_n and we have exactly $\binom{m}{2}$ such subgraphs that are edge disjoint. It follows that

$$\mathrm{sd}\,(K_n) \geq \binom{m}{2} \mathrm{sd}\,(K_{m,m})$$

As $n \to \infty$, we have $m \to \infty$ and so we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{\mathrm{sd}\,(K_n)}{\binom{n}{2}} &\geq \lim_{m\to\infty} \binom{m}{2} \frac{\mathrm{sd}\,(K_{m,m})}{\binom{n}{2}} \\
&= \lim_{m\to\infty} \binom{m}{2} \frac{m^2}{\binom{n}{2}} \\
&= \lim_{m\to\infty} \frac{m^2}{2} \frac{m^2}{\frac{n^2}{2}} \\
&= \lim_{m\to\infty} \frac{m^4}{n^2} \\
&= 1
\end{aligned}$$

Since $\mathrm{sd}\,(K_n) \leq \binom{n}{2}$, the conclusion follows.

Our result on the subdivision number of large complete bipartite graphs can easily be extended to complete n-partite graphs.

Theorem 6. *Let G denote the complete n-partite graph $K_{m_1,m_2,\ldots,m_n}$. Then*

$$\lim_{m_1,m_2,\ldots,m_n\to\infty} \frac{\mathrm{sd}\,(G)}{\sum_{i\neq j} m_i m_j} = 1.$$

Proof. For each pair $\{i, j\}$ of distinct subscripts i and j, the complete bipartite graph K_{m_i,m_j} is a subgraph of $G = K_{m_1,m_2,\ldots,m_n}$. The $\binom{n}{2}$ such subgraphs are clearly edge-disjoint. It follows that

$$\begin{aligned}
\mathrm{sd}\,(G) &\geq \sum_{i\neq j} \mathrm{sd}\,(K_{m_i,m_j}) \\
\frac{\mathrm{sd}\,(G)}{\sum_{i\neq j} m_i m_j} &\geq \frac{\sum_{i\neq j} \mathrm{sd}\,(K_{m_i,m_j})}{\sum_{i\neq j} m_i m_j} \\
&= \frac{\sum_{i\neq j} m_i m_j \frac{\mathrm{sd}\,(K_{m_i,m_j})}{m_i m_j}}{\sum_{i\neq j} m_i m_j}
\end{aligned}$$

Since $\lim_{m_i,m_j\to\infty} \frac{\mathrm{sd}\,(K_{m_i,m_j})}{m_i m_j} = 1$, the theorem follows.

In this paper, we considered the problem of finding the minimum number of edges of a graph to be subdivided in order to obtain a unit-distance graph. Our main result is that for a complete k-partite graph where $k > 1$, this minimum number is asymptotic to the size (number of edges) of the graph provided each partite set has cardinality that tends to infinity.

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On a Triangle with the Maximum Area in a Planar Point Set

Kiyoshi Hosono[1], Ferran Hurtado[2,⋆], Masatsugu Urabe[1,⋆⋆], and Jorge Urrutia[3,⋆⋆⋆]

[1] Department of Mathematics, Tokai University, Shizuoka, 424-8610 Japan

[2] Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Pau Gargallo 5, 08028-Barcelona, España

[3] Instituto de Matemáticas, Universidad Nacional Autónoma de México, México D.F., C.P.04510, México

Abstract. For a planar point set P in general position, we study the ratio between the maximum area of an empty triangle with vertices in P and the area of the convex hull of P.

1 Introduction

Let P_n be a point set with n elements in general position in the plane, $n \geq 3$. For $Q \subseteq P_n$ denote the area of the convex hull of Q by $A(Q)$. We evaluate the ratio between the maximum area of an empty triangle T with vertices in P_n and the whole area $A(P_n)$. Namely, let

$$f(P_n) = \max_{T \subset P_n} \frac{A(T)}{A(P_n)}$$

and define $f(n)$ as the minimum value of $f(P_n)$ over all point sets P_n in general position. The next result proved in [5] will be used in the proof of Theorem 1.

Theorem A. Let B be a compact convex body in the plane and B_k be a largest area k-gon inscribed in B. Then $area(B_k) \geq area(B) \frac{k}{2\pi} \sin \frac{2\pi}{k}$, where equality holds if and only if B is an ellipse.

For point sets P_n in convex position (that is when the elements of P_n are the vertices of a convex polygon) the value $f^{\mathrm{conv}}(n)$ is defined in a similar way. The following lemmas are proved in [1].

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Lemma A. For point sets in convex position with five elements

$$f^{\text{conv}}(5) = \frac{1}{\sqrt{5}}.$$

Lemma B. For point sets in convex position with six elements

$$f^{\text{conv}}(6) = \frac{4}{9}.$$

2 Points in Convex Position

We first study the value $f^{\text{conv}}(n)$. In what follows, a triangle with vertices x, y, z will be denoted by $\triangle xyz$.

Lemma 1. *Let $r(n)$ be the value of $f(P_n)$, where P_n denotes the set of vertices of a regular n-gon. Then*

$$r(n) = \frac{3\sqrt{3}}{2n\sin\frac{2\pi}{n}} \quad \text{when} \quad n \equiv 0 \bmod 3;$$

$$r(n) = \frac{2}{n} \cdot \frac{\sin\frac{\lfloor n/3\rfloor 2\pi}{n}}{\sin\frac{2\pi}{n}} \left(1 - \cos\frac{\lfloor n/3\rfloor 2\pi}{n}\right) \quad \text{when} \quad n \equiv 1 \bmod 3;$$

$$r(n) = \frac{2}{n} \cdot \frac{\sin\frac{\lceil n/3\rceil 2\pi}{n}}{\sin\frac{2\pi}{n}} \left(1 - \cos\frac{\lceil n/3\rceil 2\pi}{n}\right) \quad \text{when} \quad n \equiv 2 \bmod 3.$$

Proof. Suppose that the maximum area triangle ABC with vertices in P_n divides the boundary of the convex hull of P_n into three chains AB, BC and CA, with p, q and r edges, respectively (Fig. 1, left). We show first that any two of these numbers differ at most by 1. Suppose that this is not the case, and that for r and q we have $r - q \geq 2$. Consider the $\triangle ABD$ where D is the point symmetric to C with respect to the bisector of AB (Fig. 1, right). Assume w.l.o.g. that the line AB is horizontal. Observe that $\triangle ABC$ and $\triangle ABD$ have the same area, therefore the line DC is parallel to the line AB. Observe that since $r - q \geq 2$, there is some vertex E of P_n in the arc CD, strictly above CD. Then the area of $\triangle ABE$ is greater than the area of ABC, contradicting the choice of $\triangle ABC$.

Therefore, we conclude that the maximal area triangle splits the boundary into three chains whose numbers of edges are $\{t, t, t\}$, $\{t, t, t+1\}$, $\{t, t+1, t+1\}$, when $n \equiv 0, 1, 2 \bmod 3$, respectively. An easy computation now leads to the claimed formulas. □

Notice that each $r(n)$ is a decreasing function. Thus we can deduce that

$$\lim_{n\to\infty} r(n) = \frac{3\sqrt{3}}{4\pi}.$$

By using Theorem A and Lemma B, we obtain:

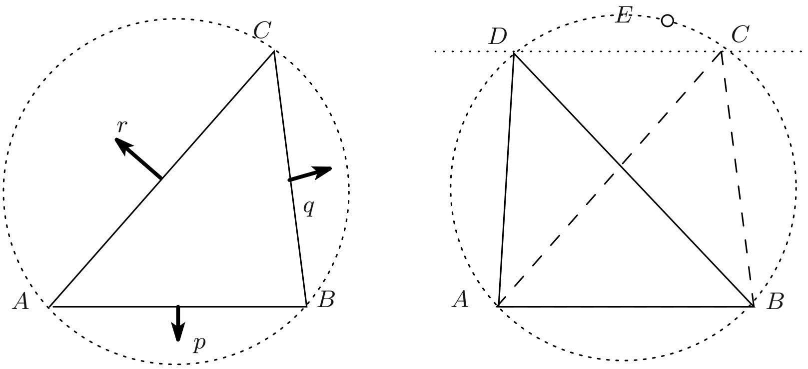

Fig. 1

Theorem 1. *For convex point sets in the plane of size $n > 6$ we have*

$$\frac{3\sqrt{3}}{4\pi} \leq f^{\text{conv}}(9) \leq \frac{4}{9} \quad \text{and} \quad \frac{3\sqrt{3}}{4\pi} \leq f^{\text{conv}}(n) \leq r(n) \ \forall n > 6, n \neq 9.$$

3 Points in General Position

In this section, we estimate the value $f(n)$ for point sets in general position. A k-hole of a point set P_n is a subset $S \subset P_n$ with k elements such that the interior of the convex hull of S does not contain any element of P_n. To prove our results, we recall the well-known theorem of Harborth [2].

Theorem B. Any planar point set with 10 or more elements has a 5-hole.

We first prove the following lemma which will be useful to determine the lower bound of $f(n)$. A k-hole is said to be non-overlapping with another l-hole if these convex hulls have disjoint interiors.

Lemma 2. *Any planar point set P_{25} with 25 elements has non-overlapping three 5-holes, or one 5-hole and one 6-hole.*

Proof. Let a, b, c be three points on the plane. Let $C(a; b, c)$ denote the convex cone with apex a determined by a, b, c. We label the elements of P_{25} from p_1 to p_{25} as follows: Let p_1 be the element of P_{25} with the smallest x-coordinate. Label the remaining points $p_2, \ldots, p_{25}$ such that the slope of the line segment joining p_i to p_1 is smaller than that of the line joining p_j iff $i < j$; $1 < i, j$.

If $C(p_1; p_2, p_{17})$ contains two non-overlapping 5-holes, we are done since $C(p_1; p_{17}, p_{25})$ has one 5-hole by Theorem B. Assume otherwise. By Theorem B each of $C(p_1; p_2, p_{10})$ and $C(p_1; p_9, p_{17})$ contains a 5-hole, call them H_1 and H_2 respectively. It follows that p_{10} is a vertex of H_1 and p_9 is a vertex of H_2. Two cases arise. H_1 contains p_1 or it does not. In the first case let $\{p_1, a, b, c, p_{10}\}$ be the vertices of H_1 (labeled in the anti-clokcwise order), and consider the domain

$D = C(p_1; p_{10}, p_{17}) \cap C(c; p_1, p_{10})$. If $D \cap P_{25}$ were empty H_1 would not overlap H_2. Then we can find a point u in D such that $\triangle p_1 p_{10} u$ is empty. Therefore there exist a 6-hole in $C(p_1; p_2, p_{17})$.

Suppose then that H_1 does not contain p_1 and label the vertices of H_1 $\{p_{10}, a, b, c, d\}$ in the anti-clockwise order. Let L be the line trough p_1 and p_{10}. Rotate L in the anti-clockwise direction around p_{10} until it meets a point q in $\{p_2, \ldots, p_{25}\}$. If q is a or an interior point of $C(p_{10}; p_1, a)$, the closed half-plane determined by the line through q and p_{10} containing p_1 has precisely 18 points and we can find two convex cones whose apex is q, each containing 7 interior points. Each contains a 5-hole. Thus P_{25} contains three non-overlapping 5-holes. Suppose then that L meets a point $q \in \{p_{11}, \ldots, p_{25}\}$. Note that $C(p_{10}; p_1, q)$ contains precisely 14 interior points. Consider the point q' such that $C(p_{10}; q, q')$ contains 7 interior points. Then if q' is contained in $C(a; p_1, p_{10})$, we are done since both $C(p_{10}; q, q')$ and $C(p_{10}; q', a)$ contain at least 7 interior points each. Otherwise, let w be the point in $C(p_{10}; q, q')$ such that $C(p_{10}; w, q')$ is empty. Then both $C(p_{10}; d, w)$ and $C(p_{10}; w, p_1)$ also contain at least 7 interior points. □

Now we can prove:

Theorem 2. *Let $n \geq 25$ be an integer. Then:*

$$\frac{23}{(37 + 3\sqrt{5})n - (97 + 6\sqrt{5})} \leq f(n) \leq \frac{1}{n-1}\,.$$

Proof. For point sets in convex position our lower bound holds trivially. Assume that P_n is not in convex position and $A(P_n) = 1$. Assume w.l.o.g. that p_1 is in the interior of the convex hull of P_n and that p_1 is the origin. Relabel the elements of $P_n - \{p_1\}$ by $p_2, \ldots, p_n$ such that for $1 < i < j$ the angle formed by the vector p_i with the x axis is smaller than that formed by p_j with the x axis. Consider the subsets $S_k = \{p_1, p_{2+23k}, p_{3+23k}, \ldots, p_{25+23k}\}$ of P, $k = 0, \ldots, \lfloor \frac{n-2}{23} \rfloor - 1$. By Lemma 2, each S_k has three non-overlapping 5-holes, or one 5-hole and one 6-hole.

Let

$$l(n) = \frac{23}{(37 + 3\sqrt{5})n - (97 + 6\sqrt{5})}$$

and assume first that each S_k has three 5-holes. If any of these 5-holes has area greater than or equal to $l(n)\sqrt{5}$, by using Lemma A, we are done.

Triangulate each of these $3\lfloor \frac{n-2}{23} \rfloor$ 5-holes, and take a triangulation T of P_n that uses these triangles. Observe that T has at most $M(n) = (2n-5) - 9\lfloor \frac{n-2}{23} \rfloor$ triangles not contained in any of the $3\lfloor \frac{n-2}{23} \rfloor$ 5-holes of P_n.

Since each 5-hole of P_n has area smaller than $l(n)\sqrt{5}$ the total area of such 5-holes is at most $3\lfloor \frac{n-2}{23} \rfloor l(n)\sqrt{5} = L(n)$. Then at least one of the $M(n)$ triangles of T not contained in the 5-holes of P_n has area greater than or equal to

$$\frac{1 - L(n)}{M(n)} \leq l(n).$$

We claim that the value obtained when some subsets S_k of P_n have both a 5-hole and a 6-hole is larger than this lower bound. For instance, we can show by using Lemma B that the bound $l(n)$ obtained for the case in which each S_k has a 5-hole and a 6-hole is $\frac{92}{(165+4\sqrt{5})n-(422+8\sqrt{5})}$.

To prove the upper bound we construct the following configuration of n points. Take an equilateral triangle with vertices $\{u, v, w\}$ of area 1 and take a point x in this triangle such that the triangles with vertices $\{u, v, x\}$ and $\{w, x, v\}$ have area $\frac{1}{n-1}$. We place now $n-4$ points $p_5, \ldots, p_n$ on the line segment xw so that they divide the line segment xw in $n-3$ intervals of the same length. Note that each triangle with vertices $\{u, p_i, p_{i+1}\}$ has area $\frac{1}{n-1}$, $i = 4, \ldots, n$ where $p_4 = x$ and $p_{n+1} = w$. Next, move slightly $p_5, \ldots, p_n$ so that $\{v, p_4, \ldots, p_{n+1}\}$ are in convex position as shown in Fig.2. Then there is no empty triangle of area greater than $\frac{1}{n-1}$ in this configuration.

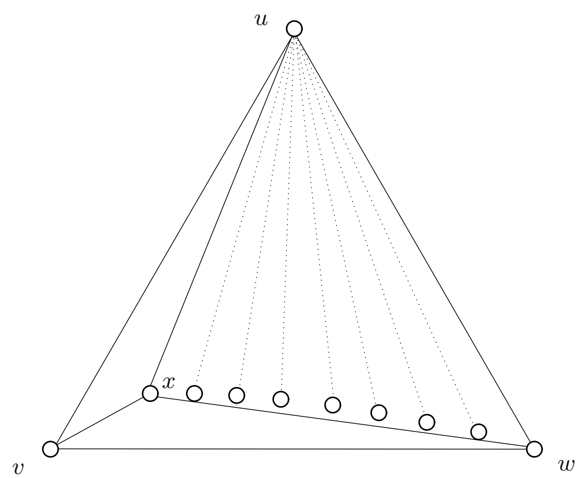

Fig. 2. The configuration to realize the upper bound □

Notes

1. If we define $f_d(n)$ in a similar way for d-dimensional Euclidean space, we can prove that

$$\frac{1}{dn - d^2 - d + 1} \leq f_d(n) \leq \frac{1}{(d-1)n - d^2 + 3}.$$

2. The problem studied in this paper is somehow related to the famous Heilbronn triangle problem: to place n points in a square of unit area so as to maximize the area of the smallest triangle determined by the n points. It has been proved that there is always a triangle of area $O(1/(n^{8/7-\epsilon}))$ and that there are point sets in which every triangle has area $\Omega(\log n/n^2)$ ([3],[4]). It has been conjectured that the later value should be the correct one.

Would the number of "small" triangles proven to be very large, one might expect to find as a consequence some "large" empty triangle, yet we have not seen so far whether this approach is feasible.

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A Balanced Interval of Two Sets of Points on a Line

Atsushi Kaneko[1] and M. Kano[2]

[1] Department of Computer Science and Communication Engineering, Kogakuin University, Nishi-Shinjuku, Shinjuku-ku, Tokyo, 163-8677, Japan
[2] Department of Computer and Information Sciences, Ibaraki University, Hitachi, Ibaraki, 316-8511, Japan
kano@cis.ibaraki.ac.jp
http://gorogoro.cis.ibaraki.ac.jp

Abstract. Let n, m, k, h be positive integers such that $1 \leq n \leq m$, $1 \leq k \leq n$ and $1 \leq h \leq m$. Then we give a necessary and sufficient condition for a configuration with n red points and m blue points on a line to have an interval containing precisely k red points and h blue points.

1 Introduction

In this paper we shall prove the following theorem:

Theorem 1. *Let n, m, k, h be integers such that $1 \leq n \leq m$, $1 \leq k \leq n$ and $1 \leq h \leq m$. Then for any n red points and m blue points on a line in general position (i.e., no two points lie on the same position.), there exists an interval that contains precisely k red points and h blue points if and only if*

$$\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)(h-1) < m < \left(\left\lfloor \frac{n-1}{k-1} \right\rfloor\right)(h+1), \tag{1}$$

where the rightmost term is an infinite number when $k = 1$.

Before giving proofs, let us give an example and explain results related to our theorem. Consider a configuration consisting of 10 red points and 20 blue points on a line in general position. Then by the above theorem, we can easily show that if $k \in \{1, 2, 3, 5, 10\}$, then such a configuration has an interval containing exactly k red points and $2k$ blue points; otherwise (i.e., $k \in \{4, 6, 7, 8, 9\}$) there exist a configuration that has no such an interval (Fig. 1). We call an interval that contains given number of red points and blue points *a balanced interval.*

Let n and $m = \lambda n$ be positive integers, where $\lambda = \frac{m}{n}$ is a rational number. Suppose that there are n red points and λn blue points in the plane in general position. Then for given point P in the plane and for any integer $1 \leq k \leq n$, there exist two rays r_1 and r_2 emanating from P such that one of the open regions determined by r_1 and r_2 contains precisely k red points and $\lfloor \lambda k \rfloor$ blue points [1] (Fig. 2 (a)) . If we choose a point P infinitely far away, then two rays

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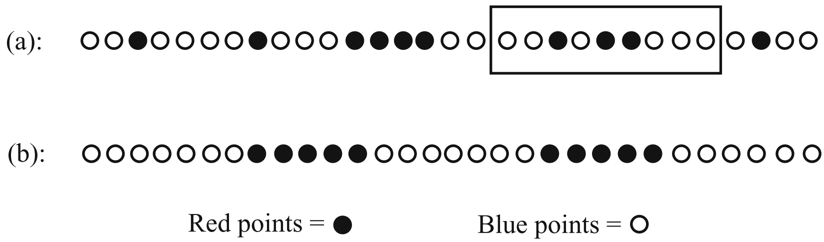

Fig. 1. (a): An interval containing 3 red points and 6 blue points; (b): A configuration that has no interval containing exactly 4 red points and 8 blue points

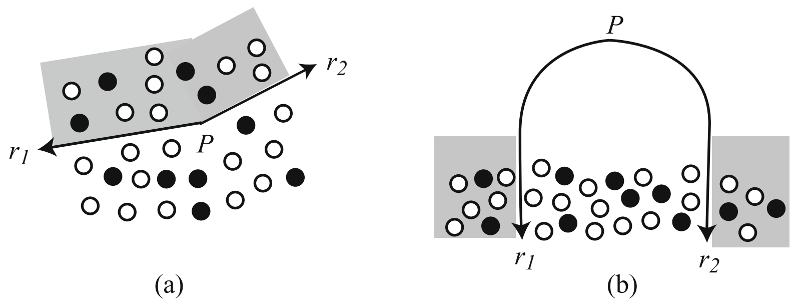

Fig. 2. (a): An open region containing 4 red points and 8 blue points; (b): A disconnected region containing 4 red points and 8 blue points when P is far away

become parallel lines, and two open regions become one connected convex region and one disconnected region consisting of two half planes ((Fig. 2 (b)). We want to know when we can always find a connected convex region determined by two parallel lines with given direction that contains precisely k red points and $\lfloor \lambda k \rfloor$ blue points. By considering the orthogonal projections of the red and blue points in the plane onto the line orthogonal to the direction, then our theorem gives an answer to this question. Other results on red and blue points in the plane can be found in [3], and definitions and notation not defined here are given in [2].

2 Proof

For a configuration with red and blue points on a line, we denote by R and B the sets of red points and blue points, respectively. A configuration X with n red points and m blue points on a line is expressed as

$$\{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_{n+m}\},$$

where each x_i denotes a red point or a blue point ordered from left to right. The configuration X is also expressed as

$$R(1) \cup B(1) \cup \cdots \cup R(s) \cup B(s),$$

where $R(i)$ and $B(i)$ denote disjoint subsets of R and B, respectively, and some of them may be empty sets. For a set Y, we denote by $|Y|$ the cardinality of Y.

We shall prove the following five lemmas, where n, m, k and h denote integers given in the theorem, that is, they satisfy $1 \le n \le m$, $1 \le k \le n$ and $1 \le h \le m$.

Lemma 1. *If*

$$m \le \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1 \right)(h-1), \tag{2}$$

then there exists a configuration with n red points and m blue points that has no interval containing exactly k red points and h blue points.

Proof. Let $t = \lfloor \frac{n}{k+1} \rfloor$. Then $m \le (t+1)(h-1)$ by (2). Hence we can construct a configuration with n red points and m blue points as follows:

$$B(1) \cup R(1) \cup B(2) \cup R(2) \cup \cdots \cup B(t+1) \cup R(t+1),$$

where $|B(i)| \le h-1$ for every $1 \le i \le t+1$, $|B(1) \cup \cdots \cup B(t+1)| = m$, $|R(i)| = k+1$ for every $1 \le i \le t$, $|R(t+1)| = n-(k+1)t \ge 0$ and $|R(1) \cup \cdots \cup R(t+1)| = n$. Then this configuration obviously has no interval containing exactly k red points and h blue points since every interval containing h blue points must include $R(j)$ for some $1 \le j \le t$, which implies the interval contains at least $k+1$ red points.

Lemma 2. *If*

$$m > \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1 \right)(h-1), \tag{3}$$

then every configuration with n red points and m blue points has an interval containing exactly k red points and at least h blue points.

Proof. Let $t = \lfloor \frac{n}{k+1} \rfloor$. Let X be a configuration with n red points and m blue points. Suppose that X has no desired interval. Namely, we assume that every interval containing exactly k red points has at most $h-1$ blue points.

Let $r_1, r_2, \cdots, r_n$ be the red points of X ordered from left to right. For integers $1 \le i < j \le n$, let $I(i,j)$ denote an open interval (r_i, r_j), and let $B(i,j)$ denote the set of blue points contained in $I(i,j)$. Furthermore, $B(-\infty, i)$ denotes the set of blue points contained in the open interval $(-\infty, r_i)$, and $B(i, \infty)$ is defined analogously. Then for any integer $1 \le s \le t-1$, $I(s(k+1), (s+1)(k+1))$ contains exactly k red points $\{r_j \mid s(k+1)+1 \le j \le (s+1)(k+1)-1\}$, and thus $|B(s(k+1), (s+1)(k+1))| \le h-1$ by our assumption. Similarly, an open interval $(-\infty, r_{k+1})$ contains exactly k red points, and thus $|B(-\infty, k+1)| \le h-1$. Moreover, since $n < (t+1)(k+1)$, $I(t(k+1), \infty)$ has at most k red points, and thus $B(t(k+1), \infty) \le h-1$. Therefore

$$\begin{aligned} |B| &\le |B(-\infty, k+1) \cup B(k+1, 2(k+1)) \cup \cdots \cup B(t(k+1), \infty)| \\ &\le (t+1)(h-1). \end{aligned}$$

This contradicts (3). Consequently the lemma is proved.

Lemma 3. *If $2 \leq k$ and*

$$m \geq \left\lfloor \frac{n-1}{k-1} \right\rfloor (h+1), \tag{4}$$

then there exists a configuration with n red points and m blue points that has no interval containing exactly k red points and h blue points.

Proof. Let $t = \lfloor \frac{n-1}{k-1} \rfloor$, which implies $t(k-1)+1 \leq n \leq (t+1)(k-1)$, and $m \geq t(h+1)$ by (4). Hence we can obtain the following configuration with n red points and m blue points:

$$R(1) \cup B(1) \cup \cdots \cup R(t+1) \cup B(t+1),$$

where $|R(i)| \leq k-1$ for every $1 \leq i \leq t+1$, $|R(1) \cup \cdots \cup R(t+1)| = n$, $|B(i)| = h+1$ for every $1 \leq i \leq t$, $|B(t+1)| = m-(h+1)t \geq 0$ and $|B(1) \cup \cdots \cup B(t+1)| = m$. Then this configuration obviously has no interval containing exactly k red points and h blue points since every interval containing k red points must include $B(j)$ for some $1 \leq j \leq t$.

Lemma 4. *If $2 \leq k$ and*

$$m < \left\lfloor \frac{n-1}{k-1} \right\rfloor (h+1), \tag{5}$$

then every configuration with n red points and m blue points has an interval containing exactly k red points and at most h blue points.

Proof. Let $t = \lfloor \frac{n-1}{k-1} \rfloor$. Then $t(k-1)+1 \leq n$. Let X be a configuration with n red points and m blue points. Suppose that X has no desired interval. Namely, we assume that every interval containing exactly k red points has at least $h+1$ blue points.

Let $r_1, r_2, \cdots, r_n$ be the red points of X ordered from left to right. For integers $1 \leq i < j \leq n$, let $I[i,j]$, denote a closed interval $[r_i, r_j]$, and let $B'(i,j)$ denote the set of blue points contained in $I[i,j]$.

Then for any integer $0 \leq s \leq t-2$, $I[k+s(k-1), k+(s+1)(k-1)]$ contains exactly k red points $\{r_j \mid k+s(k-1) \leq j \leq k+(s+1)(k-1))\}$, and thus $|B'(k+s(k-1), k+(s+1)(k-1))| \geq h+1$ by our assumption. Similarly, we have $|B'(1,k)| \geq h+1$. Therefore

$$\begin{aligned} |B| &\geq |B'(1,k) \cup B'(k, k+(k-1)) \cup \cdots \\ &\quad \cup B'(k+(t-2)(k-1), k+(t-1)(k-1))| \\ &\geq t(h+1). \end{aligned}$$

This contradicts (5). Consequently the lemma is proved.

Lemma 5. *Consider a configuration with n red points and m blue points on a line. Suppose that there exists two intervals I and J such that both I and J contain exactly k red points respectively, I contains at most h blue points, and that J contains at least h blue points. Then there exists an interval that contains exactly k red points and h blue points.*

Proof. If the sets of red points contained in I and J, respectively, are the same, then the lemma immediately follows. Thus we may assume that $I \cap R \neq J \cap R$, where R denote the set of n red points. Without loss of generality, we may assume that the leftmost red point of I lies to the left of J.

We shall show that we can move I to J step by step in such a way that the number of red points is a constant k and the number of blue points changes ± 1 at each step. We first remove the blue points left to the leftmost red point of I one by one, and then add the consecutive blue points lying to the right of I one by one, and denote the resulting interval by I_1 (Fig. 3). We next simultaneously remove the leftmost red point of I_1 and add the red point lying to the right of I_1, and get an interval I_2, which also contains exactly k red points and whose blue points are the same as those in I_1 (Fig. 3). By repeating this procedure, we can get an interval whose red point set is equal to that of J. Therefore, we can move I to J in the desired way. Consequently, we can find the required interval, which contains exactly k red points and h blue points.

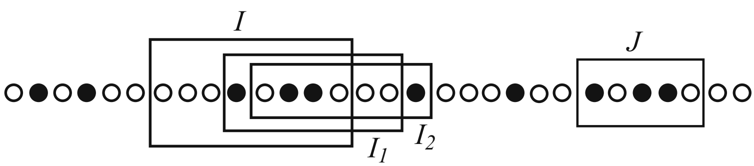

Fig. 3. Intervals $I,\ I_1,\ I_2,\ J$ containing exactly three red points

Proof of Theorem. It is obvious that every configuration with n red points and m blue points on the line has an interval containing exactly one red point and no blue point. Hence the conclusion of Lemma 4 always holds for $k = 1$. By Lemmas 1-5, the theorem follows immediately.

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Spanning Trees of Multicoloured Point Sets with Few Intersections

J. Leaños[1], C. Merino[2,⋆], G. Salazar[3,⋆⋆], and J. Urrutia[2,⋆⋆⋆]

[1] Facultad de Ciencias, UASLP, San Luis Potosí, Mexico
[2] Instituto de Matemáticas, UNAM, Mexico
[3] Instituto de Investigación en Comunicación Óptica, UASLP, San Luis Potosí, Mexico

Abstract. Kano et al. proved that if $P_0, P_1, \ldots, P_{k-1}$ are pairwise disjoint collections of points in general position, then there exist spanning trees $T_0, T_1, \ldots, T_{k-1}$, of $P_0, P_1, \ldots, P_{k-1}$, respectively, such that the edges of $T_0 \cup T_1 \cup \cdots \cup T_{k-1}$ intersect in at most $(k-1)n - k(k-1)/2$ points. In this paper we show that this result is asymptotically tight within a factor of 3/2. To prove this, we consider *alternating* collections, that is, collections such that the points in $P := P_0 \cup P_1 \cup \cdots \cup P_{k-1}$ are in convex position, and the points of the P_i's alternate in the convex hull of P.

1 Introduction

Throughout this paper we consider collections $\{P_0, P_1, \ldots, P_{k-1}\}$ of point sets in the plane. Our interest lies in the following question: what is the minimum number of intersections among the edges of a collection $\{T_0, T_1, \ldots, T_{k-1}\}$ of spanning trees for $\{P_0, P_1, \ldots, P_{k-1}\}$, respectively?

In order to avoid unnecessary complications, it makes sense to assume that our collections P_i satisfy certain properties. It is pointless to consider the case in which some P_i are empty. Similarly, having two different P_i's with nonempty intersection, or having that $\cup_{i=0}^{k-1} P_i$ is not in general position leads to pathological situations. With these observations in mind, we arrive to the following definition.

Definition. *A collection of $\{P_0, P_1, \ldots, P_{k-1}\}$ of point sets in the plane is* good *if (i) each P_i is nonempty; (ii) the P_i's are pairwise disjoint; and (iii) $\cup_{i=0}^{k-1} P_i$ is in general position.*

Let $\{P_0, P_1, \ldots, P_{k-1}\}$ be a good collection of point sets in the plane. A *corresponding set of trees for $\mathcal{P}$* is a collection $\mathcal{T} = \{T_0, T_1, \ldots, T_{k-1}\}$ such that T_i is a spanning tree for P_i, for $i = 0, \ldots, k-1$.

Given a set of trees $\{T_0, T_1, \ldots, T_{k-1}\}$, its *intersection number* $\mathrm{int}(\{T_0, T_1, \ldots, T_{k-1}\})$ is the total number of pairwise intersections of edges in $T_0 \cup T_1 \cup \cdots \cup T_{k-1}$.

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With this terminology, our problem of interest outlined above can be paraphrased as follows.

Question. *Let $\{P_0, P_1, \ldots, P_{k-1}\}$ be a good collection of point sets in the plane. What is the minimum* $\mathrm{int}(\{T_0, T_1, \ldots, T_{k-1}\})$ *taken over all corresponding sets of trees $\{T_0, T_1, \ldots, T_{k-1}\}$ for $\{P_0, P_1, \ldots, P_{k-1}\}$?*

This question was fully answered for the case $k = 2$ by Tokunaga [3]. However, the methods developed by Tokunaga do not seem to extend to $k > 2$.

In [2], Kano et al. gave the following general upper bound.

Theorem 1 (Kano et al.). *Let $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\}$ be a good collection of point sets in the plane, and let $n = |\cup_{i=0}^{k-1} P_i|$. Then there is a corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$ such that* $\mathrm{int}(\mathcal{T}) \leq (k-1)n - k(k-1)/2$.

Naturally, such a bound is of interest only if it is not too far from optimal. In the same paper, they also proved that, indeed, this bound is asymptotically tight up to a constant factor.

Theorem 2 (Kano et al.). *For each fixed k, the bound of Theorem 1 is asymptotically within a factor of* 2 *from the optimal solution.*

One of the highlights and main motivations of the present work is a proof that the bound in Theorem 1 is even tighter than as established in Theorem 2.

Theorem 3. *For each fixed k, the bound of Theorem 1 is asymptotically within a factor of* $3/2$ *from the optimal solution.*

This improvement is actually a straightforward consequence of an exhaustive analysis we perform on the special case in which the points in $\cup_{i=0}^{k-1} P_i$ are in convex position, and satisfy certain alternation condition (see proof after Theorem 8).

To test the tightness of Theorem 1, one needs to look for collections $\mathcal{P}$ for which the edges in *any* corresponding set of trees intersect a large number of times.

While seeking for such collections $\mathcal{P}$, it is quite natural to explore collections for which the points in $\cup_{i=0}^{k-1} P_i$ are in convex position. Moreover, it is intuitively appealing to propose that the points in $\cup_{i=0}^{k-1} P_i$ be arranged so that the points of each P_i "alternate as much as possible" with the points of the other P_j's.

Definition. *A good collection $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\}$ is alternating if the points in $\cup_{i=0}^{k-1} P_i$ are in convex position, and they can be labelled $p_0, p_1, \ldots, p_{sk-1}$, so that they appear in this cyclic order in the convex hull of $\cup_{i=0}^{k-1} P_i$ and, moreover, $P_i = \{p_i, p_{i+k}, \ldots, p_{i+(s-1)k}\}$, for $i = 0, 1, \ldots, k-1$.*

Note that, in particular, if $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\}$ is alternating then $|P_0| = |P_1| = \cdots = |P_{k-1}|$.

In [2], Kano et al. considered alternating collections, and described constructions of corresponding sets of trees for every $k \geq 3$. These constructions yield the following.

Proposition 1 (Kano et al. [2]). *Let $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\}$ be an alternating collection, where $k \geq 3$ and $n := |\cup_{i=0}^{k-1} P_i| \geq 2k$. Then there is a corresponding set of trees $\mathcal{T}_c$ for $\mathcal{P}$ such that* $\mathrm{int}(\mathcal{T}_c) = (3k^2/4 - k)((n/k) - 1) - k(k-2)/4$ *if k is even, and* $\mathrm{int}(\mathcal{T}_c) = (3(k-1)^2/4 + (k-1)/2)((n/k) - 1) - (k-1)^2/4$ *if k is odd.*

The constructions behind Proposition 1 are so natural that Kano et al. conjectured that they are best possible.

Conjecture 1 (Kano et al. [2]). Suppose that $\mathcal{P}$ satisfies the hypotheses of Proposition 1, and let $\mathcal{T}_c$ be the corresponding set of trees for $\mathcal{P}$ given by Proposition 1. Then, for any corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$, $\mathrm{int}(\mathcal{T}) \geq \mathrm{int}(\mathcal{T}_c)$.

One of the central results in this paper is the proof of Conjecture 1 for $k = 3$ (and also for $k = 4$; see Theorems 6 and 7). For $k = 3$, Proposition 1 claims that if $\mathcal{P} = \{P_0, P_1, P_2\}$ is alternating, and $n := |P_0 \cup P_1 \cup P_2| \geq 6$, then there is a corresponding set of trees $\mathcal{T}_c$ such that $\mathrm{int}(\mathcal{T}_c) = (4/3)n - 5$. Thus the following statement settles Conjecture 1 for $k = 3$. The proof is in Section 2.

Theorem 4. *Let $\mathcal{P} = \{P_0, P_1, P_2\}$ be an alternating collection such that $n := |P_0 \cup P_1 \cup P_2| \geq 6$. Then, for any corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$,* $\mathrm{int}(\mathcal{T}) \geq (4/3)n - 5$*. Thus Conjecture 1 holds for $k = 3$.*

Keeping in mind that the motivation behind Conjecture 1 was to search for collections $\mathcal{P}$ for which any corresponding set of trees has large intersection number, it is natural to ask if dropping the condition that $\mathcal{P}$ is alternating can yield still better (or at least comparable) results.

That is, is there a (non alternating) collection $\mathcal{P} = \{P_0, P_1, P_2\}$ for which *every* corresponding set of trees $\mathcal{T}$ has $\mathrm{int}(\mathcal{T}) \geq (4/3)|P_0 \cup P_1 \cup P_2| - 5$?

We pursued this question, and came out with a definite answer (see Section 3).

Theorem 5. *Let $\mathcal{P} = \{P_0, P_1, P_2\}$ be a non alternating good collection such that $n := |P_0 \cup P_1 \cup P_2| \geq 6$. Then there is a corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$ such that* $\mathrm{int}(\mathcal{T}) < (4/3)n - 5$*.*

Thus, if our interest lies (as it happens) in collections $\mathcal{P}$ such that $\mathrm{int}(\mathcal{T})$ is large for *every* corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$, then our best bet is to focus on alternating collections.

What about alternating collections with $k > 3$? Our first result in this regard is the following general statement, whose proof is in Section 4.

Theorem 6. *If Conjecture 1 holds for some k odd, then it also holds for $k + 1$.*

In combination with Theorem 4, this immediately yields the following.

Theorem 7. *Let $\mathcal{P} = \{P_0, P_1, P_2, P_3\}$ be an alternating collection such that $n := |P_0 \cup P_1 \cup P_2 \cup P_3| \geq 8$. Then, for any corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$,* $\mathrm{int}(\mathcal{T}) \geq 2n - 10$*. Thus Conjecture 1 holds for $k = 4$.* ∎

Theorem 4 also yields a nontrivial bound for alternating collections for all other values of k.

Theorem 8. *Let $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\}$ be an alternating collection such that $k \geq 3$ and $n := |\cup_{i=0}^{k-1} P_i| \geq 2k$. Then, for any corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$, $\mathrm{int}(\mathcal{T}) \geq (2/3)\big[(k-1)n\big] - 5k(k-1)/6$.*

This last statement follows from a standard counting argument from the case $k = 3$. The proof is in Section 5.

We conclude this introductory section with the observation that Theorem 8 implies the tightness of Theorem 1 we claimed in Theorem 3.

Proof of Theorem 3. It follows immediately from Theorem 8. ∎

2 Alternating Three–Coloured Collections: Proof of Theorem 4

Since $\mathcal{P}$ is alternating, we may assume that the points in $P = P_0 \cup P_1 \cup P_2$ are labelled so that $P_0 = \{p_0, p_3, \ldots, p_{3s-3}\}, P_1 = \{p_1, p_4, \ldots, p_{3s-2}\}$, and $P_2 = \{p_2, p_5, \ldots, p_{3s-1}\}$, in such a way that the points appear in the convex hull of P in the cyclic order $p_0, p_1, p_2, \ldots, p_{3s-1}$. Note that $n \geq 6$ implies $s \geq 2$.

We proceed by induction on s. The proof for $s = 2$ is straightforward.

Thus, we assume that the statement is true for $s = t - 1$, where $t \geq 3$, and consider the case $s = t$.

Let $\{T_0, T_1, T_2\}$ be a corresponding set of trees for $\{P_0, P_1, P_2\}$. Our aim is to show that the edges in $T_0 \cup T_1 \cup T_2$ intersect at least $(4/3)(3s) - 5 = 4s = 5$ times.

A vertex in P_i is an *i–vertex*. An edge in T_i is an *i–edge*.

A *crossing* is an intersection of edges in $T_0 \cup T_1 \cup T_2$.

Note that if every edge in $T_0 \cup T_1 \cup T_2$ has at least 3 crossings, then the total number of intersections in $T_0 \cup T_1 \cup T_2$ is at least $3(3s)/2 > 4s - 5$. By relabelling the points in $\mathcal{P}$ if necessary (perhaps even reversing the cyclic order of the points in $\mathcal{P}$), we may assume that some 0–edge e_0 has at most 2 crossings, and, moreover, that the vertices incident with e_0 are p_0 and p_{j_0}, with $j_0 \geq 6$.

It is readily checked that connectivity considerations (of the trees T_i) imply that every 0–edge intersects at least one 1–edge and at least one 2–edge. We therefore conclude that one of the crossings of e_0 occurs with a 1–edge e_1, and the other one with a 2–edge e_2. Let p_{i_1}, p_{j_1} (respectively p_{i_2}, p_{j_2}) denote the end vertices of e_1, labelled so that $0 < i_1, i_2 < j_0$ and $j_0 < j_1, j_2 < 3t - 1$.

A crossing is *internal* if both edges involved in it belong to $\{e_0, e_1, e_2\}$. A crossing is *external* if it is not internal and it involves an edge in $\{e_0, e_1, e_2\}$. A crossing is *good* if it is either internal or external.

The following statement shows that, in order to take care of the inductive step, it suffices to prove that e_0, e_1, e_2 are involved in a sufficiently large number of crossings.

Claim. In order to deal with the inductive step, it suffices to show that at least one of the following conditions holds.

(i) e_0, e_1, e_2 are incident with leaf vertices that appear consecutively in $\mathcal{P}$, and there are at least 4 good crossings.
(ii) There are at least 5 good crossings.
(iii) There are at least 4 good crossings, and e_1, e_2 are both incident with leaf vertices.

Proof. Suppose that (i) holds. Remove e_0, e_1, and e_2, and the consecutive leaves in $\mathcal{P}$ that are incident with these edges. This removes at least 4 crossings, by assumption. The result is a collection with $3(s-1)$ points to which the inductive hypothesis can be applied, to obtain at least $4(s-1)-5 = 4s-9$ crossings. These $4s-9$ crossings, together with the 4 crossings previously removed, yield at least $4s-5$ crossings in $T_0 \cup T_1 \cup T_2$, thus completing the inductive step.

Suppose now that (ii) holds. Suppose first that neither e_1 nor e_2 is incident with a leaf vertex. Remove e_1 and e_2. This removes at least 5 crossings, since e_0 by assumption only crosses e_1 and e_2. If we now contract e_0 (along with all its incident edges), collapsing p_0 and p_{j_0} and replacing them by a vertex placed in any point of e_0, we obtain two separate nonempty collections of points, of sizes $3s'$ and $3s''$, with $s'+s''=s$, to which the inductive hypothesis can be applied. This yields at least $(4s'-5)+(4s''-5) = 4s-10$ crossings, which together with the 5 crossings previously identified, give the $4s-5$ crossings required to complete the inductive step.

Now suppose that either e_1 or e_2 is incident with a leaf vertex. It is readily checked that then both e_1 and e_2 are incident with leaf vertices, and, moreover, that $p_{j_0}, p_{j_1}, p_{j_2}$ appear consecutively in $\mathcal{P}$ (moreover, $j_2 = 3t-1$) , so that p_{j_1} is the leaf vertex incident with e_1 and p_{j_2} is the leaf vertex incident with e_2. Hence in this case we might as well assume that (iii) holds. Thus we complete the proof by analyzing the case in which (iii) holds.

Suppose finally that (iii) holds. Remove e_1 and e_2. This removes at least 4 crossings, since e_0 only crosses e_1 and e_2. By contracting e_0 (along with all its incident edges), collapsing p_0 and p_{j_0} and replacing them by a vertex placed in any point of e_0, we obtain a collection of points, of size $3(s-1)$ to which the inductive hypothesis can be applied. This yields at least $4(s-1)-5 = 4s-9$ crossings, which together with the 4 crossings previously removed, give the $4s-5$ crossings required to complete the inductive step. This completes the proof of Claim 2. ∎

From Claim 2, it is clear that in order to establish the inductive step we need to show that e_0, e_1, e_2 are involved in sufficiently many crossings. Our next statement shows that a large number of crossings is always guaranteed if the end vertices of e_0, e_1, e_2 appear in certain order.

Claim. Suppose that either (a) p_0, p_{i_2}, p_{i_1} appear in $\mathcal{P}$ in the given order; or (b) $p_{j_0}, p_{j_2}, p_{j_1}$ appear in $\mathcal{P}$ in the given order. Then there are at least 3 external crossings.

Proof. We prove the statement under the assumption that (a) holds. The proof for the case in which (b) holds is totally analogous.

Since p_{i_2} cannot be an immediate successor of p_0, it follows that there is some 1–vertex p_{ℓ_1} such that $0 < \ell_1 < i_2$. Similarly, there is some 0–vertex p_{ℓ_0} such that $i_2 < \ell_0 < i_1$, and there is some 2–vertex p_{ℓ_2} such that $i_1 < \ell_2 < j_0$.

The spanning property of T_1, and the assumption that no 1–edge other than e_1 crosses e_0, imply that there is a T_1–path from p_{ℓ_1} to p_{i_1}. This path must clearly contain an edge (a 1–edge different from e_1) that crosses e_2. This provides an external crossing. Similar arguments show that some 0–edge different from e_0 must cross either e_1 or e_2, (this provides another external crossing), and that some 2–edge different from e_2 must cross e_1 (this provides a third external crossing). This completes the proof of Claim 2. ∎

We are finally ready to establish the inductive step. We analyze separately two cases, depending on whether or not e_1 and e_2 cross each other.

Case 1. *If e_1 and e_2 do not cross each other, then the inductive step follows.*

By (ii) in Claim 2, it suffices to show that there are at least 5 good crossings.

Suppose first that $i_1 < i_2$. It is readily checked that in this case the assumption that e_1 and e_2 do not cross each other implies that $p_{j_0}, p_{j_2}, p_{j_1}$ occur in this order in $\mathcal{P}$. Thus Claim 2 applies, and guarantees the required 5 good crossings (3 external crossings plus 2 internal crossings). Finally, if $i_2 < i_1$, then p_0, p_{i_2}, p_{i_1} occur in this order in $\mathcal{P}$, and again an application of Claim 2 gives the required 5 good crossings. This completes the analysis for Case 1.

Case 2. *If e_1 and e_2 cross each other, then the inductive step follows.*

We claim that, in this case, it suffices to prove the following statements:

(1) There is at least 1 external crossing.
(2) If e_0, e_1, e_2 are not incident with leaf vertices that appear consecutively in $\mathcal{P}$, then there are at least 2 external crossings.

Indeed, suppose that (1) and (2) hold. Since e_1 and e_2 cross each other, then there are 3 internal crossings. Thus, by (1), there are at least 4 good crossings. If e_0, e_1, e_2 are incident with leaf vertices that appear consecutively in $\mathcal{P}$, this fact together with (i) in Claim 2 imply that the inductive step follows. On the other hand, if e_0, e_1, e_2 are not incident with leaf vertices that appear consecutively in $\mathcal{P}$, then by (2) there are at least 5 good crossings, and so by (ii) in Claim 2 the inductive step follows.

Thus we finish the analysis of Case 2 (and thus the whole proof) by proving (1) and (2).

Before proving these statements, we make a general observation. Since e_1 and e_2 cross, then the end vertices of e_0, e_1, e_2 appear in $\mathcal{P}$ either in the order $p_0, p_{i_1}, p_{i_2}, p_{j_0}, p_{j_1}, p_{j_2}$ or in the order $p_0, p_{i_2}, p_{i_1}, p_{j_0}, p_{j_2}, p_{j_1}$. In the latter case, Claim 2 applies, in which case both (1) and (2) follow.

Therefore for proving (1) and (2) we may assume that $p_0, p_{i_1}, p_{i_2}, p_{j_0}, p_{j_1}, p_{j_2}$ appear in $\mathcal{P}$ in the given order.

One word on terminology. If p_r, p_t are vertices in $\mathcal{P}$ such that $0 \le r < t \le 3t-1$, then the *segment* $[p_r, p_t]$ is the (possibly empty) set $\{p_{r+1}, p_{r+2}, \ldots, p_{t-1}\}$.

Proof of (1)

Note that, since $j_0 \ge 6$, it follows that at least one of the segments $[p_0, p_{i_1}]$, $[p_{i_1}, p_{i_2}]$, $[p_{i_2}, p_{j_0}]$ is nonempty. Note that any such nonempty segment contains at least one 0–vertex, one 1–vertex, and one 2–vertex. Suppose for instance that $[p_{i_2}, p_{j_0}]$ is nonempty. Thus there is a 1–vertex p_{ℓ_1} such that $i_2 < \ell_1 < j_0$. The path in T_1 that joins p_{ℓ_1} and p_{j_1} must clearly cross e_2. Thus, some 1–edge other than e_1 crosses e_2. This provides an external crossing. A similar argument shows that if $[p_0, p_{i_1}]$ is nonempty, then some 2–edge other than e_2 crosses e_1. Yet another application of the same argument shows that if $[p_{i_1}, p_{i_2}]$ is nonempty, then some 0–edge other than e_0 crosses either e_1 or e_2. Therefore, in either case we obtain an external crossing, as required.

Proof of (2)

First we claim that if $p_{j_0}, p_{j_1}, p_{j_2}, p_0$ do not appear consecutively in $\mathcal{P}$, then there are at least 2 external crossings, in which case (2) immediately follows.

Suppose that $p_{j_0}, p_{j_1}, p_{j_2}, p_0$ do not appear consecutively in $\mathcal{P}$, that is, one of the segments $[p_{j_0}, p_{j_1}], [p_{j_1}, p_{j_2}], [p_{j_2}, p_0]$ is nonempty. An argument totally analogous to the one used in the proof of (1) shows that the nonemptiness of any such segments guarantees the existence of an external crossing.

Thus, if $p_{j_0}, p_{j_1}, p_{j_2}, p_0$ do not appear consecutively in $\mathcal{P}$, then there are at least two external crossings. Indeed, the crossing identified in the previous paragraph, plus the crossing obtained in the proof of (1), are clearly distinct. Thus in this case we have the required 2 external crossings.

In view of (1) and this discussion, in order to complete the proof of (2) it suffices to show the following: if $p_{j_0}, p_{j_1}, p_{j_2}, p_0$ appear consecutively in $\mathcal{P}$, and e_0, e_1, e_2 are not incident with leaf vertices that appear consecutively in $\mathcal{P}$, then then there are at least 2 external crossings. The rest of the proof is devoted to show this statement.

First we observe that since $p_{j_0}, p_{j_1}, p_{j_2}, p_0$ appear consecutively in $\mathcal{P}$, it follows that p_{j_1} and p_{j_2} are leaf vertices of e_1 and e_2, respectively. Thus the assumption that e_0, e_1, e_2 are not incident with leaf vertices that appear consecutively in $\mathcal{P}$ implies that e_0 is not incident with leaf vertices. That is, neither p_0 nor p_{j_0} is a leaf vertex.

Suppose that the segment $[p_0, p_{i_1}]$ is nonempty. Then it contains a 2–vertex, and an argument analogous to the one used in the proof of (1) shows that this implies that there is an external crossing of a 2–edge other than e_2 with e_1.

Similarly, if $[p_{i_2}, p_{j_0}]$ is nonempty, then it contains a 1–vertex, and so there is an external crossing of a 1–edge other than e_1 with e_2. By a similar token, if $[p_{i_1}, p_{i_2}]$ is nonempty, then it contains a 0–vertex, and so there is an external crossing of a 0–edge other than e_0 with either e_1 or e_2.

These arguments show that if at least two of the segments $[p_0, p_{i_1}], [p_{i_1}, p_{i_2}]$, $[p_{i_2}, p_{j_0}]$ are nonempty, then there are at least 2 external crossings, as required.

Thus for the rest of the proof we assume that exactly one of the segments $[p_0, p_{i_1}]$, $[p_{i_2}, p_{j_0}]$, $[p_{i_1}, p_{i_2}]$ is nonempty.

Suppose that $[p_0, p_{i_1}]$ is nonempty and both $[p_{i_1}, p_{i_2}]$ and $[p_{i_2}, p_{j_0}]$ are empty. Then $[p_0, p_{i_1}]$ must contain some 0–vertex. Moreover, $[p_0, p_{i_1}]$ must contain some 0–vertex that is connected to p_{j_0} via a T_0–path that does not contain p_0, as otherwise p_{j_0} would be a leaf. This implies that some 0–edge other than e_0 crosses *both* e_1 and e_2. This provides the two required external crossings.

An analogous argument takes care of the case in which $[p_{i_2}, p_{j_0}]$ is nonempty and both $[p_0, p_{i_1}]$ and $[p_{i_1}, p_{i_2}]$ are empty.

Thus we finish the proof by dealing with the case in which $[p_{i_1}, p_{i_2}]$ is nonempty and both $[p_0, p_{i_1}]$ and $[p_{i_2}, p_{j_0}]$ are empty. In this case, $[p_{i_1}, p_{i_2}]$ must contain a 0–vertex connected to p_0 via a T_0–path that does not contain p_{j_0}, as otherwise p_0 would be a leaf. For a similar reason, $[p_{i_1}, p_{i_2}]$ must contain a 0–vertex connected to p_{j_0} via a T_0–path that does not contain p_0. One of these paths must cross e_1, and the other one must cross e_2. This gives the two required external crossings. ∎

3 Non Alternating Three–Coloured Collections: Theorem 5

In an earlier version of this paper, we included a full proof of Theorem 5. However, some time after the submission, we learned about a recent related work (namely [1]), in which questions similar to, and more general than, the one explored in Theorem 5 are analyzed. In view of these new results, we decided to make this section considerably shorter, omit the proof of Theorem 5, and instead offer some remarks on the relationship between this work and the work reported in [1].

In [1], Kaneko, Kano, Suzuki, and Tokunaga analyze three–colored point sets (say R, B, and G) in *general* position, and show that the chromatic classes always have respective spanning trees with at most $2(|R| + |B|) - 4$ crossings. Thus, although our Theorem 5 is marginally better than this result for our case of interest, the result in [1] is on the other hand more general, since Kaneko, Kano, Suzuki, and Tokunaga study point sets in general (and not only convex) position.

4 Parity Issues of Conjecture 1: Proof of Theorem 6

Let $k \geq 3$ be an odd integer. We assume that Conjecture 1 holds for k, and will show that it follows that it also holds for $k+1$.

Let $\mathcal{P} = \{P_0, P_1, \ldots, P_k\}$ be an alternating collection, where $k+1$ is even (note that $\mathcal{P}$ has $k+1$ point sets P_i), and let $\mathcal{T} = \{T_0, T_1, \ldots, T_k\}$ be a corresponding set of trees.

We need to show that the edges in $T_0 \cup T_1 \cup \cdots \cup T_{k-1}$ intersect in at least $(3k^2/4 - k)((n/k) - 1) - k(k-2)/4$ points.

Since by assumption Conjecture 1 holds for k, it follows that, for each $i \in \{0,\ldots,k\}$, the edges in $(T_0 \cup \cdots \cup T_k) \setminus \{T_i\}$ intersect at least $(3(k-1)^2/4 + (k-1)/2)(((n/k)-1)-(k-1)^2/4$ times.

An elementary counting argument then shows that the edges in $T_0 \cup \cdots \cup T_k$ intersect at least $\binom{k+1}{k}\left[((3(k-1)^2/4+(k-1)/2)((n/k)-1)-(k-1)^2/4)\right]/(k-2)$ times. A straightforward manipulation shows that this number is *exactly* $(3(k+1)^2/4-(k+1))((n/k)-1)-(k+1)((k+1)-2)/4$, as required. ∎

5 Multicoloured Alternating Collections: Proof of Theorem 8

Let $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\}$ be alternating, where $k > 3$. Let $P_{r_1}, P_{r_2}, P_{r_3}$ be any three distinct collections of $\mathcal{P}$. By Theorem 4, there are at least $(4/3)(3n/k) - 5$ intersections that involve only edges in $T_{r_1} \cup T_{r_2} \cup T_{r_3}$.

Since there are $\binom{k}{3}$ ways to choose such $P_{r_1}, P_{r_2}, P_{r_3}$, an elementary counting argument shows that the total number of intersections of edges in $T_0 \cup T_1 \cup \cdots \cup T_{k-1}$ is at least

$$\binom{k}{3}\frac{(4/3)(3n/k)-5}{k-2}$$

(here we divide by $k-2$ since each P_r is in $k-2$ different 3–collections $\{P_{r_1}, P_{r_2}, P_{r_3}\}$, so that each intersection gets overcounted $k-2$ times).

A trivial manipulation shows that this expression equals $(2/3)(k-1)n - 5k(k-1)/6$, as claimed. ∎

6 Concluding Remarks

As we mentioned in Section 1, the analysis of good collections whose union is in general position is motivated by the drive to test the tightness of Theorem 1. Our Theorem 5 then shows that our best bet is to focus on alternating collections.

For alternating collections, Kano et al. put forward a general conjecture, namely Conjecture 1. In this paper we have settled this conjecture for $k = 3$ and 4. Naturally, the next step would be to try to settle the conjecture for larger values of k, aiming in the process to gain some insight into the general problem.

Proving Conjecture 1 true for larger values of k would automatically imply a better tightness estimate for Theorem 1. However, one must keep in mind that even settling Conjecture 1 for every k would not imply that Theorem 1 is (asymptotically) tight for each k. This approach to the problem of testing the tightness of Theorem 1 has a natural limit (namely a factor of 4/3), as the next result shows.

Theorem 9. *Suppose that Conjecture 1 is true for some odd integer $k_0 \geq 3$. Then, for every fixed $k \geq k_0$, the bound in Theorem 1 is asymptotically within a factor of $4k_0/(3k_0-1)$ from the optimal solution.* ∎

The proof of this statement is a straightforward counting argument.

Theorem 9 suggests that as a next step it makes sense to combine an effort to prove Conjecture 1 for $k \geq 5$ with an attempt to improve on Theorem 1. This last direction would very likely include a further exploration on the case in which the set $P_0 \cup P_1 \cup \cdots \cup P_{k-1}$ is not necessarily in convex position.

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Regular Factors Containing a Given Hamiltonian Cycle

Haruhide Matsuda*

Department of General Education, Kyushu Tokai University,
Choyo, Aso, Kumamoto, 869-1404 Japan
hmatsuda@ktmail.kokai-u.ac.jp

Abstract. Let $k \geq 1$ be an integer and let G be a graph having a sufficiently large order n. Suppose that kn is even, the minimum degree of G is at least $k+2$, and the degree sum of each pair of nonadjacent vertices in G is at least $n+\alpha$, where $\alpha = 3$ for odd k and $\alpha = 4$ for even k. Then G has a k-factor (i.e. a k-regular spanning subgraph) which is edge-disjoint from a given Hamiltonian cycle. The lower bound on the degree condition is sharp. As a consequence, we have an Ore-type condition for graphs to have a k-factor containing a given Hamiltonian cycle.

1 Introduction

We consider finite undirected graphs without loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we denote by $N_G(x)$ the neighborhood of x in G, by $\deg_G(x)$ the degree of x in G, and by $\delta(G)$ the minimum degree of G. For $S \subseteq V(G)$, let $|S|$ denote the number of the vertices in S and $G[S]$ is the subgraph of G induced by S. We write $G - S$ for $G[V(G) \setminus S]$. Given disjoint subsets $A, B \subseteq V(G)$, we write $e_G(A, B)$ for the number of edges in G joining a vertex in A to that in B.

For two integers a and b with $1 \leq a \leq b$, an $[a,b]$-*factor* is a spanning subgraph F such that $a \leq \deg_F(x) \leq b$ for each $x \in V(G)$. If $a = b = k$, then an $[a,b]$-factor is called a k-*factor*, which is a k-regular spanning subgraph.

Let us first state a well-known result which guarantees the existence of Hamiltonian cycles in a graph.

Theorem 1 (Ore [4]). *Let G be a graph of order $n \geq 3$. If for any two nonadjacent vertices x and y of G*

$$\deg_G(x) + \deg_G(y) \geq n,$$

then G has a Hamiltonian cycle.

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Theorem 1 has been generalized in several directions. Iida and Nishimura extended this theorem to that of regular factors since a Hamiltonian cycle can be regarded as a 2-factor.

Theorem 2 (Iida and Nishimura [3]). *Let $k \geq 1$ be an integer and let G be a graph of order $n \geq 4k-5$. Suppose that kn is even, $\delta(G) \geq k$, and*

$$\deg_G(x) + \deg_G(y) \geq n$$

for each pair of nonadjacent vertices x and y of $V(G)$. Then G has a k-factor.

We can easily show that if G satisfies the conditions in Theorem 2, then G has both a Hamiltonian cycle and a k-factor. However, we do not know whether these two spanning subgraphs are edge-disjoint or not. In fact, under the same degree conditions, G has a $[k, k+1]$-factor which is edge-disjoint from a given Hamiltonian cycle, but there exists a graph having no k-factor which is edge-disjoint from a given Hamiltonian cycle.

Theorem 3 (Cai, Li, and Kano [1], Matsuda [7]). *Let $k \geq 1$ be an integer and let G be a graph of order $n \geq 3$ with $n \geq 8k$ for even n and with $n \geq 6k-1$ for odd n. Suppose that $\delta(G) \geq k+2$ and*

$$\deg_G(x) + \deg_G(y) \geq n$$

for each pair of nonadjacent vertices x and y of $V(G)$. Then G has a $[k, k+1]$-factor which is edge-disjoint from a given Hamiltonian cycle.

Our purpose is to show the existence of "a k-factor which is edge-disjoint from a given Hamiltonian cycle" or "a k-factor containing a given Hamiltonian cycle". To see this, we show the following theorem.

Theorem 4. *Let $k \geq 1$ be an integer and let G be a graph of order $n > 8k^2 + 2(4-\alpha)k - \alpha$, where $\alpha = 3$ for odd k and $\alpha = 4$ for even k. Suppose that kn is even and $\delta(G) \geq k+2$. If for any two nonadjacent vertices x and y of G*

$$\deg_G(x) + \deg_G(y) \geq n + \alpha,$$

then G has a k-factor which is edge-disjoint from a given Hamiltonian cycle.

Remark 1. The lower bound on the degree condition (4) is best possible in the sense that we cannot replace $n+\alpha$ by $n+\alpha-1$, which is shown in the following example: Let $P_{t-1} = (v_1 v_2 \ldots v_{t-1})$ be a path of order $t-1$ and let $C_{t-\alpha+4} = (u_1 u_2 \ldots u_{t-\alpha+4} u_1)$ be a cycle of order $t-\alpha+4$. Then the join $G = P_{t-1} \oplus C_{t-\alpha+4}$ satisfies $n = 2t-\alpha+3$, kn is even, and

$$\frac{n+\alpha}{2} > \delta(G) = t+1 = \frac{n+\alpha-1}{2}.$$

For the Hamiltonian cycle $C = (v_1 v_2 \ldots v_{t-1} u_1 u_2 \ldots u_{t-\alpha+4} v_1)$, however, $G - E(C)$ has no k-factor since $k|P_{t-1}| < k|C_{t-\alpha+4}| - 2$ holds for $k \geq 3$.

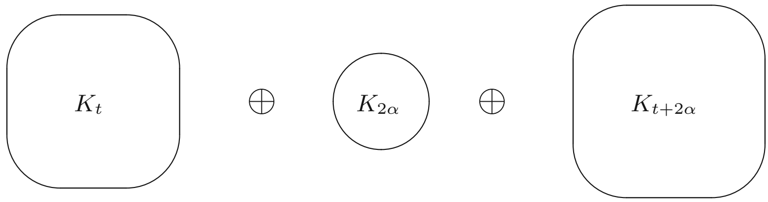

Fig. 1

Remark 2. For Theorem 4, one might consider that Theorem 2 can apply to show the existence of a regular factor in $G-E(C)$. Unfortunately, it is impossible. For example, consider the join of three complete graphs $G = K_t \oplus K_{2\alpha} \oplus K_{t+2\alpha}$, where t is a sufficiently large integer. (See Fig. 1.)

Then we find that G satisfies all conditions of Theorem 4 and that $G-E(C)$ is connected for any Hamiltonian cycle C. However, $G-E(C)$ has two independent vertices x and y of $V(K_t)$ such that

$$\begin{aligned}\deg_{G-E(C)}(x) + \deg_{G-E(C)}(y) &= 2(|K_t| + |K_{2\alpha}|) - |\{x, y\}| - |N_C(x)| - |N_C(y)| \\ &= 2(t + 2\alpha) - 6 = n - 6.\end{aligned}$$

Hence we cannot apply Theorem 2 to $G - E(C)$, but Theorem 4 guarantees that $G - E(C)$ has a k-factor.

Using Theorem 4, we obtain an Ore-type condition for graphs to have a regular factor containing a given Hamiltonian cycle.

Corollary 1. *Let $k \geq 2$ be an integer and G a graph with $n > 8k^2 - 2(\alpha + 12)k + 3\alpha + 16$, where $\alpha = 3$ for odd k and $\alpha = 4$ for even k. Suppose that kn is even and $\delta(G) \geq k$. If for any two nonadjacent vertices x and y of G*

$$\deg_G(x) + \deg_G(y) \geq n + \alpha,$$

then G has a k-factor containing a given Hamiltonian cycle.

Proof of Corollary 1. By assumption, G has a Hamiltonian cycle C. For $k = 2$, Corollary 1 holds since C itself is a desired factor. Hence we may assume that $k \geq 3$. By Theorem 4, for a given Hamiltonian cycle C, $G - E(C)$ has a $(k-2)$-factor F and hence the union $C \cup F$ is a k-factor containing a given Hamiltonian cycle, which is the desired factor. $\square$

2 Preliminary Results

Our proof for Theorem 4 depends on the following theorem, which is a special case of Tutte's f-factor theorem.

Theorem 5 (Tutte [6]). *Let G be a graph and $k \geq 1$ an integer. Then G has a k-factor if and only if*

$$\theta_G(S,T) = k|S| + \sum_{x\in T}(\deg_{G-S}(x) - k) - h_G(S,T,k) \geq 0$$

for all disjoint subsets S and T of $V(G)$, where $h_G(S,T,k)$ is the number of the components D of $G-(S\cup T)$ such that $k|D| + e_G(V(D),T)$ is odd.

We call such a component D *odd*. Let $\omega(G)$ be the number of the components of G, and let $o(G)$ be the number of the components in G each of which has odd order.

Lemma 1. *Let G be a graph satisfying the conditions of Theorem 4. Then $G - E(C)$ is connected for each Hamiltonian cycle C.*

Proof. By Theorem 1, G has a Hamiltonian cycle. For some Hamiltonian cycle C, suppose that $\omega := \omega(G-E(C)) \geq 2$. Let $C_1, C_2, \ldots, C_\omega$ be the components of $G-E(C)$. We may assume that $|C_1| \leq |C_2| \leq \cdots \leq |C_\omega|$.

If there exist two vertices $x \in V(C_1)$ and $y \in V(C_2)$ with $xy \notin E(G)$, then

$$\begin{aligned} n+\alpha \leq \deg_G(x) + \deg_G(y) &\leq |C_1| - 1 + |N_C(x)| + |C_2| - 1 + |N_C(y)| \\ &= |C_1| + |C_2| + 2, \end{aligned}$$

which implies $|C_1|+|C_2| \geq n+\alpha-2 > n$. This is a contradiction. Consequently, we may assume that all the vertices of $V(C_1)$ are adjacent to those of $V(C_2)$ in C and that $|C_1| \leq |C_2| \leq 2$. By $n > 8k^2+2(4-\alpha)k-\alpha \geq 7$ and $|C_1|+|C_2| \leq 4$, there exists a vertex $z \in V(G-(C_1\cup C_2))$ such that $xz \notin E(C)$.

By the assumption of Theorem 4,

$$\begin{aligned} n+\alpha &\leq \deg_G(x) + \deg_G(z) \\ &\leq (|C_1| - 1 + |N_C(x)|) + (n - |C_1 \cup C_2| - 1 + |N_C(y)|) \\ &= n - |C_2| + 2 < n+2, \end{aligned}$$

which contradicts $\alpha \geq 3$. □

Lemma 2. *Let G be a graph satisfying the conditions of Theorem 4 and $H = G - E(C)$ for a given Hamiltonian cycle C. Then $\omega(H-A) \leq |A| + 4 - \alpha$ for all $\emptyset \neq A \subseteq V(H)$.*

Proof. Suppose that there exists $\emptyset \neq A \subseteq V(H)$ with $\omega := \omega(H-A) \geq |A|+5-\alpha$. Since H is connected by Lemma 1, we have $\omega(H-A) \geq |A|+5-\alpha \geq 2$. Denote the components of $H-A$ by $C_1, C_2, \ldots, C_\omega$. Without loss of generality, we may assume that $|C_1| \leq |C_2| \leq \cdots \leq |C_\omega|$. If $2|A| \geq n+\alpha-4$, then $\omega \leq n-|A| \leq |A|+4-\alpha$, which is a contradiction. Thus $2|A| \leq n+\alpha-5$. By this inequality and $n \geq 8k^2+2(4-\alpha)k-\alpha+1$, we obtain $|H-A| = n-|A| \geq n-(n+\alpha-5)/2 = (n-\alpha+5)/2 \geq 4k^2+(4-\alpha)k-\alpha+3 \geq 15$. Hence there exist two vertices $x \in V(C_i)$ and $y \in V(C_j)$, $1 \leq i < j \leq \omega$, with $xy \notin E(G)$ satisfying

$$n+\alpha \leq \deg_G(x) + \deg_G(y) \leq |C_i| - 1 + |N_C(x)| + |C_j| - 1 + |N_C(y)| + 2|A|,$$

which implies $|C_i| + |C_j| \geq n - 2|A| + \alpha - 2$. Using this inequality, we have

$$n \geq |A| + \sum_{i=1}^{|A|+5-\alpha} |C_i| \geq |A| + |C_i| + |C_j| + |A| + 3 - \alpha$$
$$\geq 2|A| + (n - 2|A| + \alpha - 2) + 3 - \alpha = n + 1.$$

This is a contradiction. □

Lemma 3. *Let G be a graph satisfying the conditions of Theorem 4. Then $H = G - E(C)$ has a 1-factor for a given Hamiltonian cycle C.*

Proof. By Lemma 2, $o(H - A) \leq \omega(H - A) \leq |A| + 4 - \alpha \leq |A| + 1$. This together with $o(H - A) \equiv |A| \pmod 2$ yields $o(H - A) \leq |A|$ for all $A \subseteq V(H)$. Hence, by Tutte's 1-factor theorem [5], H has a 1-factor. □

3 Proof of Theorem 4

The proof is by contradiction. By Theorem 1, G has a Hamiltonian cycle C. Put $H = G - E(C)$. Note that $V(H) = V(G)$ and $\delta(H) = \delta(G) - 2 \geq k$.

Obviously, G has the desired factor if and only if H has a k-factor. For $k = 1$, Theorem 4 holds by Lemma 3. Hence we may consider the case $k \geq 2$. By way of contradiction, we assume that H has no k-factors. Then, by Theorem 5, there exist disjoint subsets S and T of $V(H)$ satisfying

$$\theta_H(S,T) = k|S| + \sum_{x \in T} (\deg_{H-S}(x) - k) - h_H(S,T,k) \leq -2. \tag{1}$$

We choose such subsets S and T so that $|T|$ is minimal and $|S|$ is maximal under the condition $|T|$ is minimal.

Claim 1. $|T| \geq |S| + 1$.

Proof. We first show the case when k is odd. Since H has a 1-factor by Lemma 3, H satisfies

$$|S| + \sum_{x \in T} (\deg_{H-S}(x) - 1) - h_H(S,T,1) \geq 0 \tag{2}$$

by Theorem 5. Since k is odd, $h_H(S,T,k) = h_H(S,T,1)$ holds. By subtracting (2) from (1), we have $(k-1)|T| \geq (k-1)|S| + 2$, which means $|T| \geq |S| + 1$.

Next we consider the case when k is even. Each odd component D of $H - (S \cup T)$ satisfies $k|D| + e_H(V(D),T) \equiv e_H(V(D),T) \equiv 1 \pmod 2$ and thus $e_H(V(D),T) \geq 1$. Consequently, $h_H(S,T,k) \leq \sum_{x \in T} \deg_{H-S}(x)$ holds. Substituting this inequality for (1) yields $-2 \geq \theta_H(S,T) \geq k|S| - k|T|$, which also means $|T| \geq |S| + 1$. □

Claim 2. $2|S| \leq n - 4k + \alpha - 3$.

Proof. Suppose that $2|S| \geq n - 4k + \alpha - 2$, that is, $n - 2|S| \leq 4k - \alpha + 2$. By (1) and $|T| \leq n - |S| - h_H(S,T,k)$, we obtain

$$\sum_{x\in T}\deg_{H-S}(x) \le k|T| - k|S| + h_H(S,T,k) - 2$$
$$\le k(n - |S| - h_H(S,T,k)) - k|S| + h_H(S,T,k) - 2$$
$$\le k(n-2|S|) - 2 \le k(4k-\alpha+2) - 2.$$

Since $|T| \ge |S| + 1 \ge n/2 - 2k + \alpha/2 > 4k^2 + (2-\alpha)k > 2$ by Claim 1,

$$\frac{\sum_{x\in T}\deg_{H-S}(x)}{|T|-2} \le \frac{k(4k-\alpha+2)-2}{|T|-2}$$
$$\le \frac{k(4k-\alpha+2)-2}{|S|-1} \le \frac{2k(4k-\alpha+2)-4}{n-4k+\alpha-4} < 1.$$

Note that the last inequality follows from $n > 8k^2 + 2(4-\alpha)k - \alpha$. Hence

$$\sum_{x\in T}\deg_{H-S}(x) \le |T| - 3. \tag{3}$$

Let $T_0 := \{x \in T \mid \deg_{H-S}(x) = 0\}$. It follows from Claim 1 that $n \ge |S| + |T| \ge 2|S| + 1$. Note that $n \ge 2|S| + 2$ if n is even. By the definition of α and the assumption kn is even, $n \ge 2|S| + 2$ if $\alpha = 3$ and $n \ge 2|S| + 1$ if $\alpha = 4$. Since $|T_0| \ge 3$ holds by (3), there exist two vertices x, $y \in T_0$ with $xy \notin E(G)$. Hence we have

$$n + \alpha \le \deg_G(x) + \deg_G(y) \le 2|S| + |N_C(x)| + |N_C(y)| = 2|S| + 4 < n + \alpha,$$

which is a contradiction. □

Claim 3. For any $x \in T$, $\deg_{H-S}(x) \le k - 2 + h_H(S,T,k)$ and $\deg_T(x) \le k - 2$.

Proof. Let $T' = T \setminus \{x\}$ for any $x \in T$. By the choice of T and by parity, we have $\theta_H(S,T') \ge 0$ and $\theta_H(S,T) \le -2$. Thus $2 \le \theta_H(S,T') - \theta_H(S,T) \le k - \deg_{H-S}(x) + h_H(S,T,k) - h_H(S,T',k)$, implying $\deg_{H-S}(x) \le k - 2 + h_H(S,T,k) - h_H(S,T',k)$. This inequality together with $e_H(x, H - (S\cup T)) \ge h_H(S,T,k) - h_H(S,T',k)$ yields $\deg_T(x) = \deg_{H-S}(x) - e_H(x, H-(S\cup T)) \le k - 2$. □

Claim 4. $e_H(u,T) \le k - 2$ for any $u \in V(H) \setminus (S \cup T)$.

Proof. By the choice of S, $\theta_H(S\cup\{u\},T) \ge 0$ and $\theta_H(S,T) \le -2$. Hence we obtain $2 \le \theta_H(S\cup\{u\},T) - \theta_H(S,T) \le k - e_H(u,T)$, which implies $e_H(u,T) \le k - 2$ for any $u \in V(H) \setminus (S\cup T)$. □

Let $C_1, C_2, \ldots, C_\omega$ be the odd components of $H - (S \cup T)$, where $\omega = h_H(S,T,k)$. Without loss of generality, we may assume that $|C_1| \le |C_2| \le \cdots \le |C_\omega|$.

Claim 5. $h_H(S,T,k) \le 3$.

Proof. Suppose that $h_H(S,T,k) \geq 4$. If there exist two vertices $u_1 \in V(C_1)$ and $u_2 \in V(C_2)$ with $u_1u_2 \notin E(G)$, then by Claim 4 and the assumption of Theorem 4,

$$\begin{aligned}&n+\alpha \leq \deg_G(u_1)+\deg_G(u_2)\\&\leq (|S|+(k-2)+|C_1|-1+|N_C(u_1)|)+(|S|+(k-2)+|C_2|-1+|N_C(u_2)|)\\&=2|S|+2k-2+|C_1|+|C_2|,\end{aligned}$$

which implies $|C_1|+|C_2| \geq n-2|S|-2k+\alpha+2$. By this inequality and Claims 1 and 2,

$$\begin{aligned}n &\geq |S|+|T|+|C_1|+|C_2|+|C_3|+|C_4| \geq 2|S|+1+2(|C_1|+|C_2|)\\&\geq 2|S|+1+2(n-2|S|-2k+\alpha+2)=2n-2|S|-4k+2\alpha+5>n+1,\end{aligned}$$

which is a contradiction. Consequently, we may assume that all the vertices of $V(C_1)$ are adjacent to those of $V(C_2)$ in C and that $|C_1| \leq |C_2| \leq 2$, If $|C_2|=2$, then for each $u_1 \in V(C_1)$, $e_C(u_1, V(C_2))=2$ and thus $u_1u_3 \notin E(G)$ for each vertex $u_3 \in V(C_3)$. If $|C_2|=1$, then $u_1u_2 \in E(C)$ for $\{u_1\}=V(C_1)$ and $\{u_2\}=V(C_2)$. Thus there exists a vertex $u_3 \in V(C_3)$, such that $u_1u_3 \notin E(C)$ or $u_2u_3 \notin E(C)$. Since $|T_1|=|T_2|=1$, we may assume that $u_1u_3 \notin E(C)$. Then it follows from the argument above that $|C_1|+|C_3| \geq n-2|S|-2k+2+\alpha$, and thus

$$\begin{aligned}n &\geq |S|+|T|+|C_1|+|C_2|+|C_3|+|C_4| \geq |S|+|T|+2(|C_1|+|C_3|)\\&\geq |S|+|T|+2(n-2|S|-2k+2+\alpha)=2n-2|S|-4k+2\alpha+5>n+1,\end{aligned}$$

which is a contradiction. □

Claim 6. $G[T]$ is complete.

Proof. For any two vertices x and y in T, by Claims 2, 3, 5, and $k \geq 2$, we obtain $\deg_G(x)+\deg_G(y) \leq 2(|S|+k-2+h_H(S,T,k)+2) \leq 2(k+3+|S|) \leq n+\alpha-2k+3 < n+\alpha$. Thus $xy \in E(G)$. □

Define $m_1 = \min\{\deg_{H-S}(x) \mid x \in T\}$ and let $x_1 \in T$ be a vertex satisfying $\deg_{H-S}(x_1) = m_1$. If $m_1 \geq k+1$, then by Lemma 2, $\theta_H(S,T) \geq k|S|+\sum_{x\in T}(m_1-k)-h_H(S,T,k) \geq |S|+|T|-\omega(H-(S\cup T)) \geq -1$. This contradicts (1). Hence $m_1 \leq k$.

Claim 7. $|C_1| \geq 2$.

Proof. Suppose that $|C_1| = |\{u\}| = 1$. By Claim 4, $e_H(u,T) \leq k-2$. If $e_H(u,T)=k-2$, then $k+e_H(u,T)=2k-2 \equiv 0 \pmod 2$. This contradicts the fact C_1 is an odd component of $H-(S\cup T)$. Hence $e_H(u,T) \leq k-3$. Therefore $|S|+e_H(u,T) \geq \deg_H(u) \geq \delta(H) \geq k$, implying $|S| \geq 3$. If $|T| \leq k-1$, then by Claim 5, $-2 \geq \theta_H(S,T) \geq |S|+\sum_{x\in T}(|S|+\deg_{H-S}(x)-k)-h_H(S,T,k) \geq |S|-h_H(S,T,k) \geq 0$, a contradiction. Thus we may assume that $|T| \geq k$. Since

$G[T]$ is complete by Claim 6, we have $|E(G[T])| = |T|(|T|-1)/2$. As C is a Hamiltonian cycle, $|E(G[T]) \cap C| \le |T|-1$ holds. Consequently, we obtain

$$\sum_{x \in T} \deg_{H-S}(x) \ge 2|E(G[T]) \setminus E(C)|$$
$$\ge |T|(|T|-1) - 2(|T|-1) = (|T|-1)(|T|-2).$$

Then it follows from Claim 5 and $|S| \ge 3$ that

$$\begin{aligned}\theta_H(S,T) &\ge k|S| + (|T|-1)(|T|-2) - k|T| - h_H(S,T,k)\\ &\ge 3k + |T|^2 - (k+3)|T| + 2 - 3 = (|T|-3)(|T|-k) - 1.\end{aligned}$$

If $|T| = k$, then $\theta_H(S,T) \ge -1$, which contradicts (1). If $|T| \ge k+1$, then $|T| \ge 3$ and so $\theta_H(S,T) \ge -1$, which also contradicts (1). □

Note that by Claims 1, 3, and 6, $|S|+1 \le |T| \le \deg_T(x_1) + |\{x_1\}| + |N_C(x_1)| \le k+1$, implying $|S| \le k$ and $|T| \le k+1$. We divide into three cases.

Case 1. $h_H(S,T,k) = 3$.
Since $2 \le |C_1| \le |C_2| \le |C_3|$ hold by Claim 7, there exist two vertices $u_1 \in V(C_1)$ and $u_2 \in V(C_2)$ with $u_1u_2 \notin E(G)$. Using Claim 4, we have

$$\begin{aligned}n + \alpha &\le \deg_G(u_1) + \deg_G(u_2)\\ &\le 2|S| + 2(k-2) + |C_1| - 1 + |C_2| - 1 + |N_C(u_1)| + |N_C(u_2)|,\end{aligned}$$

which implies $|C_1| + |C_2| \ge n - 2|S| - 2k + 2 + \alpha$. Then it follows from this inequality, Claim 1, $|S| \le k$, and $n > 8k^2 + 2(4-\alpha)k - \alpha$ that

$$\begin{aligned}n &\ge |S| + |T| + |C_1| + |C_2| + |C_3| \ge 2|S| + 1 + \frac{3}{2}(|C_1| + |C_2|)\\ &\ge 2|S| + 1 + \frac{3}{2}(n - 2|S| - 2k + 2 + \alpha) = \frac{3}{2}n - |S| - 3k + 4 + \frac{3}{2}\alpha\\ &\ge \frac{3}{2}n - 4k + 4 + \frac{3}{2}\alpha > n + 1.\end{aligned}$$

This is a contradiction. □

Case 2. $h_H(S,T,k) = 2$.
If there exists $u_1 \in V(C_1)$ such that $u_1x_1 \notin E(G)$, then by Claim 4,

$$\begin{aligned}n + \alpha &\le \deg_G(u_1) + \deg_G(x_1)\\ &\le (|C_1| - 1 + k - 2 + |S| + |N_C(u_1)|) + (m_1 + |S| + |N_C(x_1)|)\\ &= |C_1| + 2k + 1 + 2|S|,\end{aligned}$$

implying $|C_1| \ge n + \alpha - 2k - 1 - 2|S|$. By Claim 1 and $|S| \le k$, we obtain $n \ge |S| + |T| + |C_1| + |C_2| \ge 2|S| + 1 + 2|C_1| \ge 2|S| + 1 + 2(n + \alpha - 2k - 1 - 2|S|) = 2n - 2|S| + 2(\alpha - 2k - 1) \ge 2n - 2k + 2(\alpha - 2k - 1)$. This contradicts

$n > 8k^2 + 2(4-\alpha)k - \alpha$. Hence $T \cup C_1 \subseteq N_G(x_1) \cup \{x_1\}$, and so $|T \cup C_1| \le |N_G(x_1) \cup \{x_1\}| \le m_1 + 1 + 2 = k + 3$.

By Claim 3, $\deg_{G-S}(x) \le \deg_{H-S}(x) + 2 \le k+2$ for all $x \in T$. Since $|S| \le k$, $|T \cup C_1| \le k+3$, $n \ge 8k^2 + 2(4-\alpha)k - \alpha + 1$, $k \ge 2$, and $|T| \le k+1$, we have $n - |S \cup T \cup C_1| \ge n - (2k+3) \ge 8k^2 + 2(3-\alpha)k - \alpha - 2 > (k+2)(k+1) + 1 \ge (k+2)|T| + 1 \ge \sum_{x \in T} \deg_{G-S}(x) + 1$. Therefore there exists a vertex $u_2 \in V(H) \setminus (S \cup T \cup C_1)$ with $e_G(u_2, T) = 0$ satisfying

$$\begin{aligned} n + \alpha &\le \deg_G(u_2) + \deg_G(x_1) \\ &\le (n - |T \cup C_1| - 1 + |N_C(u_2)|) + (m_1 + |S| + |N_C(x_1)|), \end{aligned}$$

which implies

$$|T \cup C_1| \le |S| + m_1 + 3 - \alpha \le |S| + k. \tag{4}$$

If $|S| = 0$, then $|T \cup C_1| \le k$ by (4). On the other hand, $k \le \delta(H) \le \deg_H(u_1) \le |T \cup C_1| - 1$, namely, $|T \cup C_1| \ge k+1$, a contradiction. Hence $|S| \ge 1$.

If $m_1 = k$, then it follows from $|S| \ge 1$ that $\theta_H(S,T) \ge k|S| + (m_1 - k)|T| - h_H(S,T,k) \ge k - 2$. This contradicts (1). Thus $m_1 \le k-1$. By $T \cup C_1 \subseteq N_G(x_1) \cup \{x_1\}$,

$$\begin{aligned} |T| + 2 \le |T \cup C_1| &\le \deg_{H-S}(x_1) + |\{x_1\}| + |N_C(x_1) \cap (C_1 \cup T)| \\ &\le m_1 + 1 + 2 \le k + 2, \end{aligned} \tag{5}$$

which implies $|T| \le k$. We first assume that $|T| = k$. Then (5) holds thoughout with equality, that is, $|C_1| = 2$, $m_1 = k-1$, and $|N_C(x_1) \cap (C_1 \cup T)| = 2$. By (1) and $|T| = k$, we have $-2 \ge \theta_H(S,T) \ge \sum_{x \in T}(|S| + \deg_{H-S}(x) - k) - 2$, implying $|S| + \deg_{H-S}(x) = k$ for all $x \in T$. This together with $m_1 = k-1$ implies $|S| = 1$, which contradicts $k+2 \le |T \cup C_1| \le |S| + k$ by (4) and Claim 7.

Thus we may consider only the case $|T| \le k-1$. Then $\theta_H(S,T) \ge |S| + \sum_{x \in T}(|S| + \deg_{H-S}(x) - k) - h_H(S,T,k) \ge |S| + \sum_{x \in T}(\delta(H) - k) - 2 \ge |S| - 2 \ge -1$, which contradicts (1). □

Case 3. $h_H(S,T,k) \le 1$.

By (1), we have

$$-1 \ge k|S| + \sum_{x \in T}(\deg_{H-S}(x) - k). \tag{6}$$

If $|T| \le k$, it follows from (6) that $-1 \ge k|S| + \sum_{x \in T}(\deg_{H-S}(x) - k) \ge \sum_{x \in T}(|S| + \deg_{H-S}(x) - k) \ge 0$, which is a contradiction. Thus we may consider the case when $|T| = k+1$. If $m_1 = k$, then by (6), $-1 \ge k|S| + \sum_{x \in T}(\deg_{H-S}(x) - k) \ge k|S|$, a contradiction. Thus $m_1 \le k-1$ and hence $|S| \ge k - m_1 \ge 1$. Suppose that $|S| = 1$. By Claim 3, $\deg_{G-S}(x) \le \deg_{H-S}(x) + 2 \le k+1$ for all $x \in T$. Then it follows from $|T| = k+1$ that $|U| := n - |S| - |T| = n - (k+2) >$

$8k^2 - 2(\alpha-4)k - \alpha - (k+2) > (k+1)|T| + 1 \geq \sum_{x\in T} \deg_{G-S}(x) + 1$. Therefore there exists a vertex $u_1 \in U$ with $e_G(u_1, T) = 0$ and

$$(|U| - 1 + |S| + 2) + (m_1 + |S| + 2) \geq \deg_G(u_1) + \deg_G(x_1) \geq n + \alpha,$$

which implies $|U| \geq n+\alpha-k-2-2|S| = n+\alpha-k-4$. Then $n = |S|+|T|+|U| \geq k+2+(n+\alpha-k-4) = n+\alpha-2 > n$, a contradiction. Thus we may assume that $|S| \geq 2$. Since $G[T]$ is complete by Claim 6, we have $|E(G[T])| = |T|(|T|-1)/2$. As C is a Hamiltonian cycle, $|E(G[T]) \cap C| \leq |T| - 1$ holds. Consequently, we obtain

$$\begin{aligned}\sum_{x\in T} \deg_{H-S}(x) &\geq 2|E(G[T]) \setminus E(C)| \\ &\geq |T|(|T|-1) - 2(|T|-1) = (|T|-1)(|T|-2).\end{aligned}$$

Then it follows from $|S| \geq 2$ and $|T| = k+1$ that

$$\theta_H(S,T) \geq k|S| + (|T|-1)(|T|-2) - k|T| - h_H(S,T,k) \geq 2k - 2k - 1 = -1,$$

which contradicts (1). □

Finally, Theorem 4 is proved. □

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Disjoint Edges in Topological Graphs*

János Pach[1] and Géza Tóth[2]

[1] City College, CUNY and Courant Institute of Mathematical Sciences,
New York University, New York, NY 10012, USA
pach@cims.nyu.edu

[2] Rényi Institute of the Hungarian Academy of Sciences,
H-1364 Budapest, P.O.B. 127, Hungary
geza@renyi.hu

Abstract. A topological graph G is a graph drawn in the plane so that its edges are represented by Jordan arcs. G is called *simple*, if any two edges have at most one point in common. It is shown that the maximum number of edges of a simple topological graph with n vertices and no k pairwise disjoint edges is $O\left(n \log^{4k-8} n\right)$ edges. The assumption that G is simple cannot be dropped: for every n, there exists a complete topological graph of n vertices, whose any two edges cross at most twice.

1 Introduction

A *topological graph* G is a graph drawn in the plane so that its vertices are represented by points in the plane and its edges by (possibly intersecting) Jordan arcs connecting the corresponding points and not passing through any vertex other than its endpoints. We also assume that no two edges of G "touch" each other, i.e., if two edges share an interior point, then at this point they properly cross. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We will make no notational distinction between the vertices (edges) of the underlying abstract graph, and the points (arcs) representing them in the plane.

A topological graph G is called *simple* if any two edges cross at most once. G is called *x-monotone* if (in a properly chosen (x, y) coordinate system) every line parallel to the y-axis meets every edge at most once. Clearly, every *geometric graph*, i.e., every graph drawn by straight-line edges, is both simple and x-monotone.

The extremal theory of geometric graphs is a fast growing area with many exciting results, open problems, and applications in other areas of mathematics [P99]. Most of the known results easily generalize to simple x-monotone topological graphs. For instance, it was shown by Pach and Törőcsik [PT94] that for any fixed k, the maximum number of edges of a geometric graph with n vertices

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and no k pairwise disjoint edges is $O(n)$. The special cases $k = 2$ and 3 had been settled by Perles and by Alon and Erdős [AE89], respectively. All known proofs readily generalize to simple x-monotone topological graphs. (See [T00] for a more precise statement.)

Of course, here we cannot drop the assumption that G is simple, because one can draw a complete graph so that any pair of its edges cross. However, it is possible that the above statement remains true for all simple topological graphs, i.e., without assuming x-monotonicity. The aim of the present note is to discuss this problem. We will apply some ideas of Kolman and Matoušek [KM03] and Pach, Shahrokhi, and Szegedy [PSS96] to prove the following result.

Theorem 1. *For any $k \geq 2$, the number of edges of every simple topological graph G with n vertices and no k pairwise disjoint edges is at most $Cn \log^{4k-8} n$, where C is an absolute constant.*

As an immediate consequence, we obtain

Corollary. *Every simple complete topological graph with n vertices has $\Omega(\frac{\log n}{\log \log n})$ pairwise disjoint edges.* □

The best previously known lower bound for this quantity, $\Omega\left(\log^{1/6} n\right)$, was established by Pach, Solymosi and Tóth [PST01].

We also prove that Theorem 1 does not remain true if we replace the assumption that G is simple by the slightly weaker condition that any pair of its edges cross at most *twice*.

Theorem 2. *For every n, there exists a complete topological graph of n vertices whose any pair of edges have exactly one or two common points.*

The analogous question, when the excluded configuration consists of k *pairwise crossing* (rather than pairwise disjoint) edges, has also been considered. For $k = 2$, the answer is easy: every crossing-free topological graph with $n > 2$ vertices is *planar*, so its number of edges is at most $3n - 6$. For $k = 3$, it was shown by Agarwal et al. [AAP97] that every geometric graph (in fact, every simple x-monotone topological graph) G with n vertices and no 3 pairwise crossing edges has $O(n)$ edges. This argument was extended to all topological graphs by Pach, Radoičić and Tóth [PRT03a]. It is a major unsolved problem to decide whether, for any fixed $k > 3$, every geometric (or topological) graph of n vertices which contains no k pairwise crossing edges has $O(n)$ edges. It is known, however, that the number of edges cannot exceed n times a polylogarithmic factor [PSS96], [V98], [PRT03a]. Here the assumption that G is *simple* does not seem to play such an central role as, e.g., in Theorem 1.

2 Auxiliary Results

In this section, after introducing the necessary definitions, we review, modify, and apply some relevant results of Kolman and Matoušek [KM03] and Pach, Shahrokhi, and Szegedy [PSS96].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any partition of $V(G)$ into two non-empty parts, V_1 and V_2, let $E(V_1, V_2)$ denote the set of edges connecting V_1 and V_2. The set $E(V_1, V_2) \subset E(G)$ is said to be a *cut*. The *bisection width* $b(G)$ of G is defined as the minimum size $|E(V_1, V_2)|$ of a cut with $|V_1|, |V_2| \geq |V|/3$. The *edge expansion* of G is

$$\beta(G) = \min_{V_1 \cup V_2 = V(G)} \frac{|E(V_1, V_2)|}{\min\{|V_1|, |V_2|\}},$$

where the first minimum is taken over *all* partitions $V_1 \cup V_2 = V(G)$.

Clearly, we have $\beta(G) \leq 3b(G)/n$. On the other hand, it is possible that $\beta(G)$ is small (even 0) but $b(G)$ is large. However, it is very easy to prove

Lemma 1. [KM03] *Every graph G of n vertices has a subgraph H of at least $2n/3$ vertices such that $\beta(H) \geq b(G)/n$.* □

An *embedding* of a graph H in G is a mapping that takes the vertices of H to distinct vertices of G, and each edge of H to a path of G between the corresponding vertices. The *congestion* of an embedding is the maximum number of paths passing through an edge of G.

As Kolman and Matoušek have noticed, combining a result of Leighton and Rao [LR99] for multicommodity flows with the rounding technique of Raghavan and Thompson [RT87], we obtain the following useful result.

Lemma 2. [KM03] *Let G be any graph of n vertices with edge expansion $\beta(G) = \beta$. There exists an embedding of the complete graph K_n in G with congestion $O(\frac{n \log n}{\beta})$.* □

The *crossing number* $\text{CR}(G)$ of a graph G is the minimum number of crossing points in any drawing of G. The *pairwise crossing number* $\text{PAIR-CR}(G)$ and the *odd-crossing number* $\text{ODD-CR}(G)$ of G are defined as the minimum number of pairs of edges that cross, resp., cross an *odd* number of times, over all drawings of G. It follows directly from the definition that for any graph G $\text{CR}(G) \geq \text{PAIR-CR}(G) \geq odd - cr(G)$. For any graph G, let

$$\text{SSQD}(G) = \sum_{v \in V(G)} d^2(v),$$

where $d(v)$ is the degree of the vertex v in G, and SSQD is the shorthand for the "sum of squared degrees."

Next we apply Lemmas 1 and 2 to obtain the following assertion, slightly stronger than the main result of Kolman and Matoušek [KM03], who established a similar inequality for the pairwise crossing number.

Two edges of a graph are called *independent* if they do not share an endpoint.

Lemma 3. *For every graph G, we have*

$$\text{ODD-CR}(G) \geq \Omega\left(\frac{b^2(G)}{\log^2 n}\right) - O\left(\text{SSQD}(G)\right).$$

Moreover, G has at least this many pairs of independent edges that cross an odd number of times.

Proof. Let H be a subgraph of G satisfying the condition in Lemma 1. Using the trivial inequality $\text{ODD-CR}(G) \geq \text{ODD-CR}(H)$, it is sufficient to show that

$$\text{ODD-CR}(H) \geq \Omega\left(\frac{n^2\beta^2(H)}{\log^2 n}\right) - O\left(\text{SSQD}(H)\right).$$

Letting m denote the number of vertices of H, we have $n \geq m \geq 2n/3$.

Fix a drawing of H, in which precisely $\text{ODD-CR}(H)$ pairs of edges cross an odd number of times. For simplicity, this drawing (topological graph) will also be denoted by H. In view of Lemma 2, there exists an embedding of K_m in H with congestion $O(\frac{m \log m}{\beta(H)})$. In a natural way, this embedding gives rise to a drawing of K_m, in which some portions of Jordan arcs representing different edges of K_m may coincide. By a slight perturbation of this drawing, we can obtain another one that has the following properties:

1. any two Jordan arcs cross a finite number of times;
2. all of these crossings are proper;
3. if two Jordan arcs originally shared a portion, then after the perturbation every crossing between the modified portions occurs in a very small neighborhood of some (point representing a) vertex of H.

Let e_1 and e_2 be two edges of K_m, represented by two Jordan arcs, γ_1 and γ_2, respectively. By the above construction, each crossing between γ_1 and γ_2 occurs either in a small neighborhood of a vertex of H or in a small neighborhood of a crossing between two edges of H. Therefore, if γ_1 and γ_2 cross an odd number of times, then either (i) one of their crossings is very close to a vertex of H, or (ii) γ_1 and γ_2 contain two subarcs that run very close to two edges of H that cross an odd number of times. Clearly, the number of pairs (γ_1, γ_2) satisfying conditions (i) and (ii) is at most the square of the congestion of the embedding of K_m in H multiplied by $\text{SSQD}(H)$ and by $\text{ODD-CR}(H)$, respectively. Thus, we have

$$\text{ODD-CR}(K_m) = O\left((\text{SSQD}(H) + \text{ODD-CR}(H))\frac{n^2 \log^2 n}{\beta^2(H)}\right).$$

On the other hand, it is known ([PT00]) that $\text{ODD-CR}(K_m) = \Omega(m^4)$. Comparing these two bounds and taking into account that $m \geq 2n/3$, the lemma follows. □

Theorem 3. *For any $k \geq 2$, every topological graph of n vertices that contains no k independent edges such that every pair of them cross an odd number of times, has at most $Cn \log^{4k-8} n$ edges, for a suitable absolute constant C.*

Proof. We use double induction on n and k. For $k = 2$ and for every n, the statement immediately follows from an old theorem of Hanani [H34], according to which $\text{ODD-CR}(G) = 0$ holds if and only if G is planar.

Assume that we have already proved Theorem 3 for some $k \geq 2$ and for all n. For $n = 1, 2$ the statement is trivial. Let $n > 2$ and suppose that the assertion is also true for $k + 1$ and for all topological graphs having fewer than n vertices.

We prove, by double induction on k and n, that the number of edges of a topological graph G with n vertices, which has no $k+1$ edges that pairwise cross an odd number of times, is at most $Cn\log^{4k-4} n$. Here C is a constant to be specified later. The statement is trivial for $k=1$. Suppose that it holds (1) for $k-1$ and for all n, and (2) for k and for every $n' < n$. For simplicity, the underlying abstract graph is also denoted by G. For any edge $e \in E(G)$, let $G_e \subset G$ denote the topological graph consisting of all edges of G that cross e an odd number of times. Clearly, G_e has no k edges so that any pair of them cross an odd number of times. By part (2) of the induction hypothesis, we have

$$\text{ODD-CR}(G) \leq \frac{1}{2}\sum_{e\in E(G)} |E(G_e)| \leq \frac{1}{2}|E(G)|Cn\log^{4k-8} n.$$

Using the fact that $\text{SSQD}(G) \leq 2|E(G)|n$ holds for every graph G, it follows from Lemma 3 that

$$b(G) \leq C_0 \log n \left(|E(G)|n\log^{4k-8} n\right)^{1/2},$$

for a suitable constant C_0. Consider a partition of $V(G)$ into two parts of sizes $n_1, n_2 \leq 2n/3$ such that the number of edges running between them is $b(G)$. Neither of the subgraphs induced by these parts has $k+1$ edges, any pair of which cross an odd number of times. Applying part (1) of the induction hypothesis to these subgraphs, we obtain

$$|E(G)| \leq Cn_1\log^{4k-4} n_1 + Cn_2\log^{4k-4} n_2 + b(G).$$

Comparing the last two inequalities and setting $C = \max(100, 10C_0)$, the result follows by some calculation. □

3 Proofs of the Main Results

Proof of Theorem 1. Let G be a simple topological graph with no k pairwise disjoint edges. Let G' be a *bipartite* topological subgraph of G, consisting of at least half of the edges of G, and let V_1 and V_2 denote its vertex classes.

Applying a suitable homeomorphism (continuous one-to-one transformation) to the plane, if necessary, we can assume without loss of generality that

1. all vertices in V_1 lie above the line $y=1$;
2. all vertices in V_2 lie below the line $y=0$;
3. each piece of an edge that lies in the strip $0 \leq y \leq 1$ is a vertical segment.

Replace the part of the drawing of G' that lies above the line $y=1$ by its reflection about the y-axis. Erase the part of the drawing in the strip $0 \leq y \leq 1$, and re-connect the corresponding pairs of points on the lines $y=0$ and $y=1$ by straight-line segments.

If in the original drawing two edges, $e_1, e_2 \in E(G')$, have crossed each other an *even* number of times, then after the transformation their number of crossings

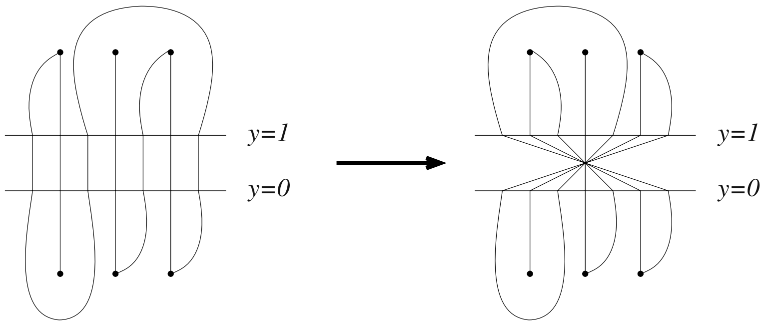

Fig. 1. The redrawing procedure

will be *odd*, and vice versa. Indeed, if originally e_i crossed the strip k_i times, then k_i was odd ($i = 1, 2$.) After the transformation, we have $k_1 + k_2$ pairwise crossing segments in the strip $0 \le y \le 1$. From the $\binom{k_1+k_2}{2}$ crossings between them, $\binom{k_i}{2}$ correspond to self-intersections of e_i. Thus, the number of crossings between e_1 and e_2 in the resulting drawing is equal to their original number of crossings plus

$$\binom{k_1 + k_2}{2} - \binom{k_1}{2} - \binom{k_2}{2}.$$

However, this sum is always odd, provided that k_1 and k_2 are odd. Note that one can easily get rid of the resulting self-intersections of the edges by locally modifying them in small neighborhoods of these crossings.

Suppose that the resulting drawing of G' has k edges, any two of which cross an odd number of times. Then any pair of the corresponding edges in the original drawing must have crossed an even number of times. Since originally G' was a *simple* topological graph, i.e., any two of its edges crossed at most once, we can conclude that the original drawing of G' (and hence the original drawing of G) had k pairwise disjoint edges, contradicting our assumption.

Thus, the new drawing of G' has no k edges that pairwise cross an odd number of times. Now it follows directly from Theorem 3 that $|E(G)| \le 2|E(G')| \le 2Cn \log^{4k-8} n$, as required. □

The assumption that G is simple was used only once, at the end of the proof. Another implication of the redrawing procedure is that Theorem 3 hold also if we replace“odd” by “even.”

Proof of Theorem 2. Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n. For $1 \le i \le n$, place v_i at $(i, 0)$. Now, for any $1 \le i < j \le n$, represent the edge v_iv_j by a polygon whose vertices are

$$(0,i),\ (0, i - j/n),\ (i - j/n - n, 0),\ (0, i - j/n - n),\ (0, j).$$

It is easy to verify that any two of these polygons cross at most twice. □

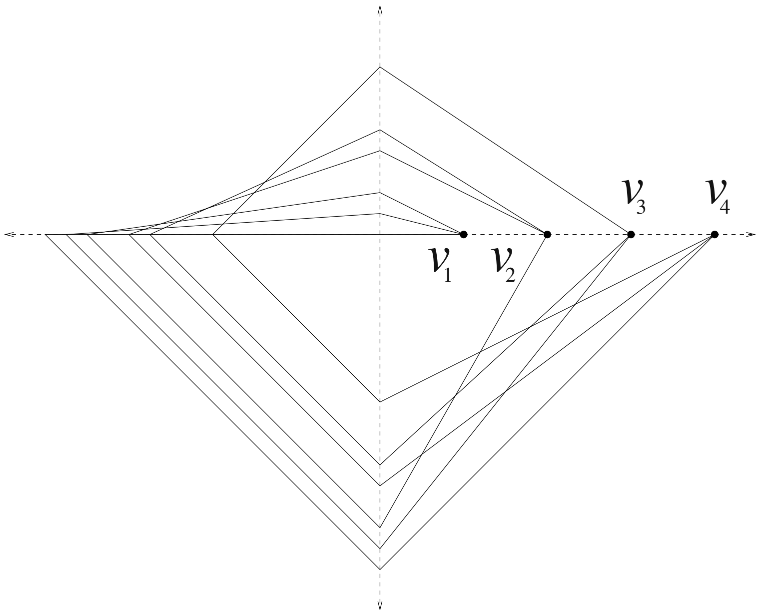

Fig. 2. A drawing of K_4 in which any two edges have exactly one or two common points

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The Decycling Number of Cubic Graphs

Narong Punnim*

Department of Mathematics, Srinakharinwirot University,
Sukhumvit 23, Bangkok 10110, Thailand
narongp@swu.ac.th

Abstract. For a graph G, a subset $S \subseteq V(G)$, is said to be a *decycling set* of G if if $G \setminus S$ is acyclic. The cardinality of smallest decycling set of G is called the *decycling number* of G and it is denoted by $\phi(G)$.

Bau and Beineke posed the following problems: Which cubic graphs G with $\mid G \mid = 2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$? In this paper, we give an answer to this problem.

Keywords: Degree sequence, decycling number, cubic graph.

1 Introduction

We consider in this paper only finite simple graphs. For the most part, our notation and terminology follows that of Bondy and Murty [2]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Since we deal only with finite and simple graphs, we will use the following notations and terminology for a typical graph G. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. As usual, we use $|S|$ to denote the cardinality of a set S and therefore we define $n = |V|$ to be the *order* of G and $m = |E|$ the *size* of G. To simplify writing, we write $e = uv$ for the edge e that connects the vertices u and v. The *degree* of a vertex v of a graph G is defined as $d(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree of a graph G is usually denoted by $\Delta(G)$. Let S and T be disjoint subsets of $V(G)$ of a graph G. We denote by $e(S, T)$ the number of edges in G that connect from S and T. If S is a subset of $V(G)$ of a graph G, by the graph S we mean the induced subgraph of S in G and we denote $e(S)$ to be the number of edges in the graph S. A graph G is said to be *regular* if all of its vertices have the same degree. A 3-regular graph is called a *cubic graph.*

Let G be a graph of order n and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is called a *degree sequence* of G. Moreover, a graph H of order n is said to have the same degree sequence as G if there is a bijection f from $V(G)$ to $V(H)$ such that $d(v_i) = d(f(v_i))$ for all $i = 1, 2, \ldots, n$. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a

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graphic degree sequence if it is a degree sequence of some graph G and in this case, G is called a *realization* of $\mathbf{d}$.

Let G be a graph, and let ab and cd be independent edges in G such that ac and bd are not edges in G. Define $G^{\sigma(a,b;c,d)}$ to be the graph obtained from G by deleting the edges ab and cd and replacing the edges ac and bd. The operation $\sigma(a,b;c,d)$ is called *switching operation.* It is easy to see that the graph obtained from G by a switching will have the same degree sequence as G. The following theorem has been shown by Havel [4] and Hakimi [3].

Theorem 1.1. *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a graphic degree sequence. If* G_1 *and* G_2 *are any two realizations of* $\mathbf{d}$*, then* G_2 *can be obtained from* G_1 *by a finite sequence of switchings.* □

As a consequence of Theorem 1.1, we can define *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$, the vertices of which are the graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. Thus, as a direct consequence of Theorem 1.1, we have shown the following theorem.

Theorem 1.2. *The graph* $\mathcal{R}(\mathbf{d})$ *is connected.* □

Let G be a graph. The problem of determining the minimum number of vertices whose removal eliminates all cycles in the graph G is difficult even for some simply defined graphs as stated in Bau and Beineke [1]. For a graph G, this minimum is known as the *decycling number* of G, and denoted by $\phi(G)$. However, the class of those graphs G in which $\phi(G) = 0$ consists of all forests, and $\phi(G) = 1$ if and only if G has at least one cycle and a vertex is on all of its cycles. It is also easy to see that $\phi(K_n) = n - 2$ and $\phi(K_{p,q}) = p - 1$ if $p \leq q$, where K_n and $K_{p,q}$ denote the complete graph of order n and the complete bipartite graph with partite sets of cardinality p and q, respectively. The value of $\phi(G)$ for many classes of the graphs were obtained by Bau and Beineke [1] including an upper bound of connected cubic graphs of a given girth. In the same paper, they posed the following problems:

Problem 1. Which cubic graphs G with $|G| = 2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$?

Problem 2. Which cubic planar graphs G with $|G| = 2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$?

We shall answer the Problem 1.

We proved in [5] that the graph parameter ϕ has the property that if the graphs G_1 and G_2 are adjacent in the graph $\mathcal{R}(\mathbf{d})$, then $|\phi(G_1) - \phi(G_2)| \leq 1$. Thus for any graphic degree sequence $\mathbf{d}$, there exist integers a and b such that there is a graph G with degree sequence $\mathbf{d}$ and $\phi(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$. We proved in the same paper that if $\mathcal{R}(3^{2n})$ is the class of all cubic graphs of order $2n$, then $\min\{\phi(G) : G \in \mathcal{R}(3^{2n})\} = \lceil \frac{n+1}{2} \rceil$. Thus to answer the Problem 1 is equivalent to find all cubic graphs of order $2n$ in $\mathcal{R}(3^{2n})$ having minimum cardinality of decycling set.

2 Main Results

For a graphic degree sequence $\mathbf{d}$, let $\phi(\mathbf{d}) = \{\phi(G) : G \in \mathcal{R}(\mathbf{d})\}$. Thus there exist integers a and b such that $\phi(\mathbf{d}) = \{k \in \mathbb{Z} : a \leq k \leq b\}$. For each $c \in \phi(\mathbf{d})$, let $\mathcal{R}(\mathbf{d};\ c)$ denote the subgraph of the graph $\mathcal{R}(\mathbf{d})$ induced by the vertices corresponding to graphs with decycling number c. We consider the problem of determining the structure of induced subgraph $\mathcal{R}(\mathbf{d};\ c)$. In general, what is the structure of $\mathcal{R}(\mathbf{d};\ c)$? In particular, are these graphs connected? If $\mathcal{R}(\mathbf{d};\ c)$ is connected, it must be possible to generate all realizations of $\mathbf{d}$ with decycling number c by beginning with one such realization and applying a suitable sequence of switchings producing only graphs with decycling number c. In this section, we find all cubic graphs of order $2n$ with decycling number $\lceil \frac{n+1}{2} \rceil$ and prove that the induced subgraph $\mathcal{R}(3^{2n};\ \lceil \frac{n+1}{2} \rceil)$ is connected.

Let G be a cubic graph of order $2n$ with a minimum decycling set S of cardinality $\lceil \frac{n+1}{2} \rceil$. Since there is only one cubic graph of order 4 and there are only two cubic graphs of order 6, it is easy to that those graphs have the decycling number $\lceil \frac{n+1}{2} \rceil$. From now on we will consider when $2n \geq 8$. Put $F = G \setminus S$. Thus $e(S) + e(S, F) + e(F) = 3n$, $2|S| \leq e(S, F) \leq 3|S|$ and $e(S, F) = 3|S| - 2e(S)$. Hence $2n - \lceil \frac{n+1}{2} \rceil - 1 \geq e(F) = 3n - 3\lceil \frac{n+1}{2} \rceil + e(S)$. Thus we have the following Lemma:

Lemma 2.1. *Let G be a cubic graph of order $2n$ with a minimum decycling set S of cardinality $\lceil \frac{n+1}{2} \rceil$. Put $F = G \setminus S$. Then*

(1) $e(S) = 0$ *if n is odd, and* $e(S) \leq 1$ *if n is even,*
(2) *if n is odd, then F is a tree,*
(3) *if n is even and $e(S) = 1$, then F is a tree,*
(4) *if n is even and $e(S) = 0$, then F has 2 connected components.* □

For an odd integer $n \geq 5$, there exists a cubic graph G with independent decycling set S of G of cardinality $\frac{n+1}{2}$. Furthermore, $F = G \setminus S$ is a tree of order $N = \frac{3n-1}{2}$ and $\Delta(F) \leq 3$. Let $n_i, i = 1, 2, 3$ be the number of vertices of F of degree i. It is clear that $n_1 + n_2 + n_3 = N$, $2n_1 + n_2 = \frac{3(n+1)}{2}$ and $n_1 \geq 2$.

Let $\mathcal{F}(n_1, n_2, n_3)$ be the class of trees F having n_i vertices of degree $i, i = 1, 2, 3$ and $\Delta(F) \leq 3$. Thus for any $F \in \mathcal{F}(n_1, n_2, n_3)$ we have $|V(F)| = n_1 + n_2 + n_3$, $n_1 = n_3 + 2$, and $n_1 \geq 2$.

Lemma 2.2. *Let n be an odd integer with $n \geq 5$ and $N = \frac{3n-1}{2}$. Then there exists a cubic graph G with $\phi(G) = \frac{n+1}{2}$ if and only if there exist nonnegative integers $n_i, i = 1, 2, 3$, $n_1 + n_2 + n_3 = N$, $2n_1 + n_2 = \frac{3(n+1)}{2}$ and $n_1 \geq 2$.*

Proof. We have already proved the first part of this lemma. Suppose there exist nonnegative integers $n_i, i = 1, 2, 3$, $n_1 + n_2 + n_3 = N$, $2n_1 + n_2 = \frac{n+1}{2}$, and $n_1 \geq 2$. We first consider when $n_1 = 2$. Thus $n_2 = \frac{3n-5}{2}$, $n_3 = 0$. Let F is a path of order N with $V(F) = F_1 \cup F_2$, where $F_1 = \{f_1, f_2\}$ and $F_2 = \{f_3, f_4, \ldots, f_N\}$ are the sets of vertices of F of degree 1 and of degree 2 of F, respectively. Let G be a graph with $V(G) = V(F) \cup S$, where $S = \{s_1, s_2, \ldots, s_{\frac{n+1}{2}}\}$, and $E(G) = \{s_1f_1, s_1f_2, s_1f_3, s_2f_1, s_2f_2, s_2f_4\} \cup E_1$, where $E_1 = \bigcup_{i=0}^{\frac{n-5}{2}} \{s_{3+i}f_{3i+5}, s_{3+i}f_{3i+6}, s_{3+i}f_{3i+7}\}$. It is clear that $G \in \mathcal{R}(3^{2n};\ \frac{n+1}{2})$.

Suppose $n_1 \geq 3$. Since $n'_1 = n_1 - 1$, $n'_2 = n_2 + 2$ and $n'_3 = n_3 - 1$ satisfy the conditions of the theorem, by induction on n_1, there exists $G' \in \mathcal{R}(3^{2n};\ \lceil \frac{n+1}{2} \rceil)$ and a minimum decycling set S such that $F' = G' - S \in \mathcal{F}(n'_1, n'_2, n'_3)$. Let $F'_i, i = 1, 2, 3$ be the corresponding vertices in F' of degree i. Since $n'_2 \geq 2$, there exists $v, w \in F'_2$ and $v \neq w$. Since v has two neighbors in F', there exists $u \in V(F')$ such that $u \neq w$, $uv \in E(F')$, and $uw \notin E(F')$. Let $s_1, s_2 \in S$ such that $vs_1, ws_2 \in E(G')$. We will consider into 2 cases.

Case 1. If $s_1 \neq s_2$, then the graph $G = G'^{\sigma(u,v;w,s_2)}$ contains a maximum induced forest F with $V(F) = V(F')$ having n_1 vertices of degree 1.

Case 2. If $s_1 = s_2$, then there exist $s \in S \setminus \{s_1\}$ and $x \in V(F') \setminus \{v, w\}$ such that $xs \in E(G')$ and $xs_1 \notin E(G')$. Thus the graph $G'^{\sigma(x,s;s_1,v)}$ contains a maximum induced forest F with $V(F) = V(F')$, v, w are of degree 2 in F, and v, w have different neighbors in S. By applying a suitable switching in Case 1., we get a cubic graph G with n_1 vertices of degree 1.

Thus the proof is complete. □

By Lemma 2.1 and the similar argument in Lemma 2.2, we obtain the following two results.

Lemma 2.3. *Let n be an even integer with $n \geq 4$ and $N = \frac{3n}{2} + 1$. Then there exists a cubic graph G of order $2n$ with minimum decycling set S such that $|S| = \frac{n}{2} + 1$ and $e(S) = 1$ if and only if there exist nonnegative integers $n_i, i = 1, 2, 3$, $n_1 + n_2 + n_3 = N$, $2n_1 + n_2 = \frac{3n}{2} + 1$ and $n_1 \geq 2$.* □

Lemma 2.4. *Let n be an even integer with $n \geq 4$ and $N = \frac{3n}{2} + 1$. Then there exists a cubic graph G_1 of order $2n$ with minimum decycling set S_1 such that $|S_1| = \frac{n}{2} + 1$ and $e(S_1) = 0$ if and only if there exist a cubic graph G of order $2n$ with a minimum decycling set S of cardinality $\frac{n+1}{2}$, $e(S) = 1$, and a suitable switching.* □

Let n, N, n_1, n_2, and n_3 be integers satisfying the conditions in Lemma 2.2. We first consider in the case when $n_1 = 2$ in the class $\mathcal{F}(n_1, n_2, n_3)$. Thus $n_3 = 0$ and $\mathcal{F}(2, n_2, 0)$ contains the path of N vertices. Let P_N be the path $f_1 f_2 \cdots f_N$ and let $S_t = \{s_1, s_2, \ldots, s_t\}$ be a set of independent vertices, where $t = \frac{n+1}{2}$. It is clear that there are cubic graphs obtained by joining $3t$ edges from S to the vertices in P_N. In particular case when $N = 4$, there is a unique cubic graph G_4 obtained in this way. That is $V(G_4) = V(S_2) \cup V(P_4)$ and $E(G_4) = \{s_1f_1, s_1f_2, s_1f_4, s_2f_1, s_2f_3, s_2f_4\}$. Let G_7 be a cubic graph with $V(G_7) = V(S_3) \cup V(P_7)$ and $E(G_7) = (E(G_4) \setminus \{s_2f_4\}) \cup \{s_2f_7, s_3f_5, s_3f_6, s_3f_7\}$. Thus G_7 is a cubic graph of order 10 with $\phi(G_7) = 3$. The graph G_{10} can be obtained from G_7 by extending the path P_7 to P_{10}, removing the edge s_3f_7 and inserting edges $s_3f_{10}, s_4f_8, s_4f_9, s_4f_{10}$. In general, if $t \geq 4$, then $N = 3t - 2$. We can construct the cubic graph G_N obtained from G_{N-3} by extending the path P_{N-3} to P_N, removing the edge $s_{t-1}f_{N-3}$ and inserting edges $s_{t-1}f_N, s_tf_{N-2}, s_tf_{N-1}, s_tf_N$. The graph G_N is called the *standard cubic graph of order $2n$*.

Lemma 2.5. *Let n be an odd integer and G a cubic graph of order $2n$ with $\phi(G) = \frac{n+1}{2}$. If G has a path P_N as a maximum induced forest, where $N = \frac{3n-1}{2}$, then G_N*

can be obtained from G by a finite number of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that for every $i = 1, 2, \ldots t$, $G^{\sigma_1\sigma_2\ldots\sigma_i}$ is a cubic graph with P_N as its induced forest.

Proof. It is easy if $N = 4$. Let G be a cubic graph of order $2n$ with P_N as its induced forest. Put $P_N = f_1 f_2 \cdots f_N$ and $S_t = \{s_1, s_2, \ldots, s_t\}$ where $t = \frac{n+1}{2}$. If $s_t f_N \notin E(G)$, there are exactly 2 vertices in S which are adjacent to f_N and there are exactly 3 vertices in $V(P_N)$ which are adjacent to s_t. Thus there exist $s_i \in S$ and $f_j \in V(P_N)$ such that $s_i f_N, s_t f_j \in E(G)$ and $s_i f_j \notin E(G)$. The graph $G^1 = G^{\sigma_1}$, where $\sigma_1 = \sigma(s_t, f_j; f_N, s_i)$, has a common edge $s_t f_N$ with G_N. If $s_t f_{N-1} \notin E(G^1)$, there exists $s \in S_t$ such that $s f_{N-1} \in G^1$. Put $s = s_{t-1}$. Since P_N is a path and $t \geq 3$, $|N(s_{t-1}) \cap N(s_t)| \leq 1$. Thus there exists $f_j \in V(P_N)$ such that $f_j \neq f_{N-1}, s_t f_j \in E(G^1)$, and $s_{t-1} f_j \notin E(G^1)$. Therefore the graph $G^2 = G^{1\sigma_2}$, where $\sigma_2 = \sigma(s_t, f_j; f_{N-1} s_{t-1}$, has $s_t f_N, s_t f_{N-1}$ as common edges with G_N. By continuing in this way, we can transform the graph G by a finite number of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ to G_N such that for every $i = 1, 2, \ldots t$, $G^{\sigma_1\sigma_2\ldots\sigma_i}$ is a cubic graph with P_N as its induced forest. □

Lemma 2.6 *Let n is an odd integer and G a cubic graph of order $2n$ with $\phi(G) = \frac{n+1}{2}$. If G has no P_N as its maximum induced forest, then G_N can be obtained from G and a finite number of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that for every $i = 1, 2, \ldots t$, $\phi(G^{\sigma_1\sigma_2\ldots\sigma_i}) = \frac{n+1}{2}$.*

Proof. By the argument in the proofs of Lemma 2.2 and Lemma 2.5 a sequence of suitable switchings can be obtained in order to transform G into G_N. □

Similar argument can be made to obtain the same result for cubic graphs of order $2n$ and n is even.

We have constructed all cubic graphs of order $2n$ having decycling set of cardinality $\frac{n+1}{2}$. In particular, we have proved the following theorem.

Theorem 2.7. *The induced subgraph $\mathcal{R}(3^{2n}; \lceil \frac{n+1}{2} \rceil)$ is connected.* □

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Equal Area Polygons in Convex Bodies

T. Sakai[1], C. Nara[1], and J. Urrutia[2,⋆]

[1] Research Institute of Educational Development, Tokai University,
2-28-4 Tomigaya, Shibuya-ku, Tokyo 151-8677, Japan
{tsakai, cnara}@ried.tokai.ac.jp

[2] Instituto de Matemáticas, Ciudad Universitaria,
Universidad Nacional Autónoma de México, México D.F., México
urrutia@matem.unam.mx

Abstract. In this paper, we consider the problem of packing two or more equal area polygons with disjoint interiors into a convex body K in $\boldsymbol{E}^2$ such that each of them has at most a given number of sides. We show that for a convex quadrilateral K of area 1, there exist n internally disjoint triangles of equal area such that the sum of their areas is at least $\frac{4n}{4n+1}$. We also prove results for other types of convex polygons K. Furthermore we show that in any centrally symmetric convex body K of area 1, we can place two internally disjoint n-gons of equal area such that the sum of their areas is at least $\frac{n-1}{\pi}\sin\frac{\pi}{n-1}$. We conjecture that this result is true for any convex bodies.

1 Introduction

For a subset S of $\boldsymbol{E}^2$ having a finite area, let $A(S)$ denote the area of S. A compact convex set with nonempty interior is called a *convex body*.

In [2], W. Blaschke showed the following theorem:

Theorem A. *Let K be a convex body in $\boldsymbol{E}^2$, and let T be a triangle with maximum area among all triangles contained in K. Then $\frac{A(T)}{A(K)} \geq \frac{3\sqrt{3}}{4\pi}$ with equality if and only if K is an ellipse.*

E. Sás [13] generalized Blaschke's result as follows:

Theorem B. *Let K be a convex body in $\boldsymbol{E}^2$, and let P be a polygon with maximum area among all polygons contained in K and having at most n sides. Then $\frac{A(P)}{A(K)} \geq \frac{n}{2\pi}\sin\frac{2\pi}{n}$ with equality if and only if K is an ellipse.*

For subsets $A_1, \cdots, A_m$ of $\boldsymbol{E}^2$, we say that the A_i are *internally disjoint* if the interiors of any two A_i and A_j with $1 \leq i < j \leq m$ are mutually disjoint. In this paper, we consider the problem of packing two or more equal area internally disjoint polygons in a convex body in $\boldsymbol{E}^2$ such that each of them has at most a given number of sides, and the sum of their areas is maximized.

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Let K be a convex body in $\boldsymbol{E}^2$ and let $\mathcal{P}_{m,n}(K)$ denote a family of m internally disjoint equal area convex polygons $P_1, \cdots, P_m \subset K$ such that each P_i, $1 \le i \le m$, has at most n sides, and define

$$s(K; m, n) = \sup_{\{P_1, \cdots, P_m\} \in \mathcal{P}_{m,n}(K)} \frac{A(P_1) + \cdots + A(P_m)}{A(K)}.$$

We simply write $t_m(K)$ for $s(K; m, 3)$. Clearly $t_m(T) = 1$ for any triangle T and positive integer m, and hence $s(T; m, n) = 1$ for any triangle T and integers $m \ge 1$ and $n \ge 3$. In general, for any integers k, m, n with $n \ge k \ge 3$, $m \ge 1$ and for any convex polygon K with at most k sides, $s(K; m, n) = 1$ (Fig. 1).

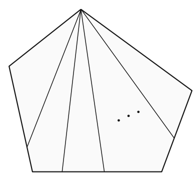


Fig. 1

Monsky [10] showed that a rectangle can be dissected into m equal area triangles if and only if m is even. Thus

Theorem C. *Let m be a positive integer and let R be a rectangle. Then $t_m(R) = 1$ for any even integer m and $t_m(R) < 1$ for any odd integer m.*

Furthermore, Kasimatis showed that a regular k-gon, $k \ge 5$, can be dissected into m equal area triangles if and only if m is a multiple of k [6]; and Kasimatis and Stein showed that almost all polygons cannot be dissected into equal area triangles [7]:

Theorem D. *Let k be an integer with $k \ge 5$ and let K be a regular k-gon. Then $t_m(K) = 1$ for any positive integer $m \equiv 0$ (mod k) and $t_m(K) < 1$ for any positive integer $m \not\equiv 0$ (mod k).*

Theorem E. *For almost all polygons K and for any integer $m \ge 1$, $t_m(K) < 1$.*

2 Preliminary Results

We now show some propositions that will be needed to prove our results. For a subset S of $\boldsymbol{E}^n$, we denote the convex hull of S by $conv(S)$.

Proposition 1. *Let n be an integer with $n \ge 3$, P a convex polygon with at least n sides, and let α denote the value of the maximum area of a convex polygon contained in P with at most n sides. Then there exists an n-gon of area α each of whose vertices is a vertex of P.*

Proof. Let $P = p_1p_2 \cdots p_k$, $k \geq n$. Take a convex polygon $Q \subseteq P$ with at most n sides such that $A(Q) = \alpha$ and the number of common vertices of P and Q is maximized. By way of contradiction, suppose that there is a vertex a of Q such that $a \notin \{p_1, \cdots, p_k\}$. By the maximality of $A(Q)$, a is on the boundary of P, and hence a is an interior point of a side of P. We may assume $a \in p_1p_2 - \{p_1, p_2\}$. Let b and c be distinct vertices of Q adjacent to a. Then $A(abc) \leq \max\{A(p_1bc), A(p_2bc)\}$. We may assume $A(abc) \leq A(p_1bc)$. Let $Q' = \mathrm{conv}((Q - abc) \cup p_1bc)$. Then $Q' \subseteq P$, Q' has at most n sides, $\alpha = A(Q) \leq A(Q')$ (so $\alpha = A(Q')$ by the maximality of α), and the number of common vertices of Q' and P is strictly greater than that of Q and P, a contradiction. Thus any vertex of Q is a vertex of P, and it follows from the maximality of α that Q has n sides. □

Proposition 2. *Let K be a convex body in $\boldsymbol{E}^2$ and let m and n be integers with $m \geq 3$ and $n \geq 3$. Suppose that K contains internally disjoint polygons $P = p_1 \cdots p_m$ and $Q = q_1 \cdots q_n$. Then K contains internally disjoint polygons P' and Q' such that* $\mathrm{conv}(P' \cup Q')$ *has at most $m + n - 2$ sides, P' has at most m sides, Q' has at most n sides, and $A(P') \geq A(P)$, and $A(Q') \geq A(Q)$.*

Remark 1. A simple proof for the case where $m = n = 3$ is shown in [12].

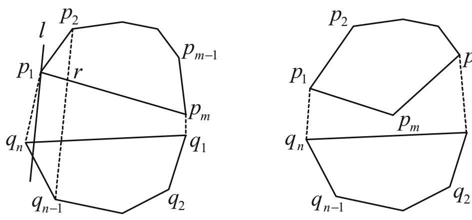
Proof. Let $S = \mathrm{conv}(P \cup Q)$. If S has at most $m+n-2$ sides, then we have only to let $P' = P$ and $Q' = Q$. Thus assume that S has $m+n$ sides or $m+n-1$ sides.

Case 1. S has $m + n$ sides:
We may assume that $S = p_1p_2 \cdots p_mq_1q_2 \cdots q_n$ and that the straight line l passing through p_1 and parallel to p_2q_{n-1} satisfies the condition that $(l \cap p_1p_2q_{n-1}q_n) - \{p_1\} \neq \emptyset$ (Fig. 2 (a)). Let r be the intersection point of p_1p_m and p_2q_{n-1}. Then $A(q_np_2r) \geq A(p_1p_2r)$, and hence $P^* = q_np_2p_3 \cdots p_m$ is a convex polygon with m sides such that P^* is internally disjoint to Q and $A(P^*) \geq A(P)$. Using the same arguments for P^* and Q, we obtain P' and Q' with the desired properties.

Case 2. S has $m + n - 1$ sides:
We may assume that $S = p_1p_2 \cdots p_{m-1}q_1q_2 \cdots q_n$ and that $A(p_1p_{m-1}q_1) \geq A(p_1p_{m-1}q_n)$ (Fig. 2 (b)). Then $A(p_1p_{m-1}q_1) \geq A(p_1p_{m-1}p_m)$, and hence $P^* = p_1p_2 \cdots p_{m-1}q_1$ is a convex polygon with m sides such that P^* is internally disjoint to Q and $A(P^*) \geq A(P)$. Proceeding the same way as in the latter part of the proof for Case 1, we obtain P' and Q' with the desired properties. □

Proposition 3. *Let $P = p_1p_2p_3p_4p_5$ be a convex pentagon with $A(P) = 1$ and let $\alpha = \frac{5-\sqrt{5}}{10}$. Then there exist indices i and j such that $A(p_{i-1}p_ip_{i+1}) \leq \alpha \leq A(p_{j-1}p_jp_{j+1})$ (indices are taken modulo 5).*

Proof. We first show that there exists an index i such that $A(p_{i-1}p_ip_{i+1}) \leq \alpha$. By way of contradiction, suppose that $A(p_{i-1}p_ip_{i+1}) > \alpha$ for any i with $1 \leq i \leq 5$. Then $A(p_1p_2p_3) > \alpha$, $A(p_1p_2p_5) > \alpha$ and $A(p_1p_2p_4) < 1 - 2\alpha$. Let q be the intersection point of p_1p_4 and p_3p_5 (Fig. 3). Since $A(p_1p_2q) \geq$



(a) (b)

Fig. 2

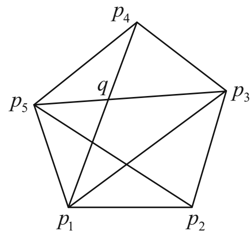


Fig. 3

$\min\{A(p_1p_2p_3),\, A(p_1p_2p_5)\} > \alpha$, $\frac{p_1q}{p_1p_4} = \frac{A(p_1p_2q)}{A(p_1p_2p_4)} > \frac{\alpha}{1-2\alpha}$. Therefore $A(p_3p_4p_5) = (1 - A(p_1p_2p_3)) \times \frac{qp_4}{p_1p_4} < (1-\alpha) \times \frac{1-3\alpha}{1-2\alpha}$. On the other hand, we have $A(p_3p_4p_5) > \alpha$ by assumption. Consequently, $\alpha < \frac{(1-\alpha)(1-3\alpha)}{1-2\alpha}$, and hence we must have $5\alpha^2 - 5\alpha + 1 > 0$. This contradicts $\alpha = \frac{5-\sqrt{5}}{10}$. Similarly we can verify that there exists an index j such that $A(p_{j-1}p_jp_{j+1}) \geq \alpha$. □

We conclude this section with two more propositions shown in [12]. Proposition 4 is obtained by using the *Ham Sandwich Theorem* (see, for example, [9, 14]) and a small adjustment, and Proposition 5 is obtained by using an extension of the Ham Sandwich Theorem shown in [1, 5, 11]:

Proposition 4. *Let n be an integer with $n \geq 3$ and let K be a convex polygon with at most n sides. Then $s(K, 2, \lfloor \frac{n}{2} \rfloor + 2) = 1$.*

Proposition 5. *Let n be an integer with $n \geq 3$ and let K be a convex polygon with at most n sides. Then $s(K, 3, \lceil \frac{n}{3} \rceil + 4) = 1$.*

Remark 2. Combining Propositions 4 and 5, we obtain several results. For example, for a convex polygon K with at most $k = 2^l + 3$ sides, we have $s(K; 1, 2^l + 3) = s(K; 2, 2^{l-1} + 3) = s(K; 2^2, 2^{l-2} + 3) = \cdots = s(K; 2^l, 4) = s(K; 2^{l+1}, 4) = s(K; 2^{l+2}, 4) = \cdots = 1$ (and $s(K; 2^{l+1}, 3) \geq \frac{8}{9}$ by the equality $s(K; 2^l, 4) = 1$ and Theorem 2 to be shown in Section 3); for a polygon K with at most $k = 3^l + r$ sides, $r \in \{6, 7\}$, we have $s(K; 1, 3^l + r) = s(K; 3, 3^{l-1} + r) = s(K; 3^2, 3^{l-2} + r) = \cdots = s(K; 3^l, 1 + r) = s(K; 3^{l+1}, 7) = s(K; 3^{l+2}, 7) = \cdots = 1$; for a polygon with at most 30 sides, $s(K; 3, 14) = s(K; 6, 9) = s(K; 12, 6) = 1$; and so on.

3 Equal Area Polygons in a Convex Polygon

Theorem 1. *Let K be a convex body in $\boldsymbol{E}^2$ and let $\boldsymbol{u}$ be a non-zero vector in $\boldsymbol{E}^2$. Then there exist internally disjoint equal area triangles T_1 and T_2 in K such that $T_1 \cap T_2$ is a segment parallel to $\boldsymbol{u}$ and $A(T_1) + A(T_2) \geq \frac{1}{2}A(K)$.*

Proof. Let l_1 and l_2 be distinct straight lines, each of which is parallel to $\boldsymbol{u}$ and tangent to K (Fig. 4). Let a be a contact point of l_1 and K and let b be a contact point of l_2 and K. Let m be the midpoint of the segment ab, and let c and d be intersection points of the perimeter of K and the straight line passing through m and parallel to $\boldsymbol{u}$. Let e and g be the intersection points of the straight line tangent to K at c and straight lines l_1 and l_2, respectively, and let f and h be the intersection points of the straight line tangent to K at d and straight lines l_1 and l_2, respectively. Let l_3, l_4 be straight lines perpendicular to $\boldsymbol{u}$ and passing through c, d, respectively, and label the vertices of the rectangle surrounded by l_1, l_2, l_3 and l_4, as shown in Fig. 4. Then for triangles $T_1 = acd$ and $T_2 = bcd$, $T_1 \cap T_2$ is a segment parallel to $\boldsymbol{u}$, $A(T_1) = A(T_2)$, and it follows from the convexity of K that

$$A(T_1) + A(T_2) = \frac{1}{2}A(e'g'h'f') = \frac{1}{2}A(eghf) \geq \frac{1}{2}A(K),$$

as desired. □

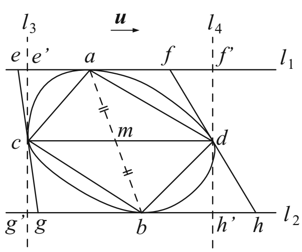


Fig. 4

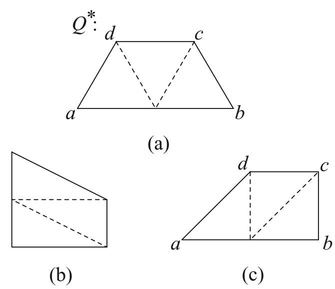


Fig. 5

Theorem 2. *Let K be a convex quadrilateral. Then the following hold:*

(i) $t_2(K) \geq 9/8$ with equality if and only if K is affinely congruent to the quadrilateral Q^* shown in Fig. 5 (a); and
(ii) $t_n(K) \geq 4n/4n + 1$ for any integer $n \geq 2$.

Proof. (i) Let $K = p_1p_2p_3p_4$. We may assume

$$A(p_1p_2p_4) \geq A(p_1p_2p_3) \text{ and } A(p_1p_2p_4) \geq A(p_1p_3p_4). \tag{1}$$

By considering a suitable affine transformation f, we may assume further that $f(p_1) = O(0,0)$, $f(p_2) = a(1,0)$, $f(p_4) = c(0,1)$ (Fig. 6 (a)). Write $f(p_3) = b$, let $e = (1,1)$ and let m be the midpoint of ac. By (1) and the convexity of K, $b \in ace$. By symmetry, we may assume that $b \in ame$. Let d be the intersection point of the straight lines Om and bc. Then d is on the side bc and $A(Oad) = A(Ocd)$. We show that $A(Oad) + A(Ocd) \geq 8/9A(K)$. For this purpose, we let b' be the intersection point of the straight lines bc and $x = 1$ (Fig. 6 (b)), and we show that $2A(Oad) \geq 8/9A(Oab'c)$.

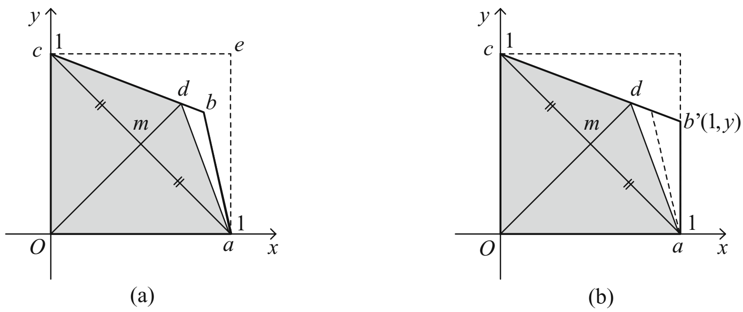


Fig. 6

Write $b' = (1, y)$. We have $0 < y \leq 1$, $A(Oab'c) = \frac{y+1}{2}$. Furthermore, since $d = \left(\frac{1}{2-y}, \frac{1}{2-y}\right)$, $2A(Oad) = \frac{1}{2-y}$. Hence,

$$\frac{2A(Oad)}{A(Oab'c)} = \frac{2}{(2-y)(y+1)} = \frac{2}{-\left(y-\frac{1}{2}\right)^2 + \frac{9}{4}} \geq \frac{8}{9},$$

as desired.

Next we show that for a convex quadrilateral K, $t_2(K) = \frac{8}{9}$ holds if and only if K is affinely congruent to Q^*. If $t_2(K) = \frac{8}{9}$, then, in the argument above, we must have $b = b'$ and $y = \frac{1}{2}$. Hence $t_2(K) = \frac{8}{9}$ implies that K is affinely congruent to the quadrilateral shown in Fig. 5 (b), and hence to Q^*. Now we show that for a convex quadrilateral K affinely congruent to Q^* and for any choice of two internally disjoint equal area triangles T_1 and T_2 in K, $\frac{A(T_1)+A(T_2)}{A(K)} \leq \frac{8}{9}$. It suffices to show this for the case where $K = Q^*$, whose vertices are labeled as shown in Fig. 5 (a). Let T_1 and T_2 be internally disjoint equal area triangles in K, and let l be a straight line such that each of the half-planes H_1 and H_2 with $H_1 \cap H_2 = l$ contains one of T_1 or T_2. Let p and q be the intersection points of l and the perimeter of K. Four cases arise:

(a) $\{p, q\} \in ab \cup bc$ or $\{p, q\} \in ab \cup da$;
(b) $\{p, q\} \in ab \cup cd$;
(c) $\{p, q\} \in bc \cup cd$ or $\{p, q\} \in cd \cup da$;
(d) $\{p, q\} \in bc \cup da$.

First consider case (b). We may assume $p \in ab, q \in cd$ and $T_1 \subseteq apqd$. Write $S = A(apq) = A(apd)$ and $S' = A(aqd) = A(pqd)$. By Proposition 1, $A(T_1) \leq S$ or $A(T_1) \leq S'$. If $A(T_1) \leq S$, then we can retake T_1 in apd, and this case is reduced to Case (a). If $A(T_1) \leq S'$, then we can retake T_1 in aqd, and this case is reduced to Case (c). Next consider Case (c). By symmetry, we consider only the case when $\{p, q\} \in bc \cup cd$. We may assume $p \in bc, q \in cd$ and $T_1 \subseteq cpq$. Then $A(T_1) \leq A(bcd) \leq \frac{1}{3}A(K)$. Hence $A(T_1) + A(T_2) \leq \frac{2}{3}A(K) < \frac{8}{9}A(K)$ in this case. Next consider Case (d). We may assume $p \in bc, q \in da$ and $T_1 \subseteq abpq$. By symmetry, we may assume further that $bp \geq aq$. Then since $A(T_1) \leq A(abp)$, we can retake T_1 in abp, and hence this case is reduced to Case (a).

Finally, we consider Case (a). By symmetry, we consider only the case where $\{p, q\} \in ab \cup bc$. Furthermore, by considering a suitable affine transformation, we may assume that $K = abcd$ is the trapezoid shown in Fig. 5 (c) with $bc = 1$, $p \in ab, q \in bc$ and $T_1 \subseteq bpq$. We show that $A(T_1) + A(T_2) \leq \frac{8}{9}A(K) = \frac{4}{3}$. For this purpose, we suppose that $A(T_1) \geq \frac{2}{3}$ and show that $A(T_2) \leq \frac{2}{3}$. In view of Proposition 1, it suffices to show that any triangle whose vertices are in $\{a, p, q, c, d\}$ has area at most $\frac{2}{3}$. Let $x = bp$ and let $y = bq$. Since $\frac{2}{3} \leq A(T_1) \leq A(bpq)$ by assumption, $x \geq \frac{4}{3}$ and $y \geq \frac{2}{3}$. Hence $A(cdq) \leq A(cdp) \leq A(qdp) \leq A(qda) = A(abcd) - (A(abq) + A(cdq)) = 1 - \frac{y}{2} \leq \frac{2}{3}$, $A(acd) = \frac{1}{2}$, $A(apq) \leq A(apc) = A(apd) = \frac{2-x}{2} \leq \frac{1}{3}$ and $A(cpq) \leq A(caq) = 1 - y \leq \frac{1}{3}$. Thus we have $A(T_2) \leq \frac{2}{3}$, as desired.

(ii) Let $K = p_1p_2p_3p_4$. We may assume that $A(K) = 1$ and $A(p_1p_2p_3) \geq \frac{1}{2}$. We show (ii) by induction on n. Suppose that $t_n(K) \geq \frac{4n}{4n+1}$ for some $n \geq 2$. Take point q on p_2p_3 such that $A(p_1p_2q) = \frac{4}{4(n+1)+1}\ \left(< \frac{1}{2}\right)$. By induction, there exist n internally disjoint triangles $T_1, \cdots, T_n$ in $p_1qp_3p_4$ such that $A(T_1) = \cdots = A(T_n) = \frac{4}{4n+1} \times A(p_1qp_3p_4) = \frac{4}{4(n+1)+1} = A(p_1p_2q)$. Thus $t_{n+1}(K) \geq \frac{4(n+1)}{4(n+1)+1}$, as desired. □

Theorem 3. *Let K be a convex pentagon. Then the following hold:*

(i) $t_2(K) \geq \frac{2}{3}$*;*
(ii) $t_3(K) \geq \frac{3}{4}$*; and*
(iii) $t_n(K) \geq \frac{2n}{2n+1}$ *for any integer* $n \geq 4$*.*

Proof. Let $K = p_1p_2p_3p_4p_5$. We may assume that

$$A(p_1p_2p_5) \geq A(p_ip_{i+1}p_{i+2}) \quad \text{for} \quad 1 \leq i \leq 4, \tag{2}$$

where $p_6 = p_1$.
(i) By considering a suitable affine transformation f, we may assume that $f(p_1) = O(0, 0)$, $f(p_2) = a(1, 0)$, $f(p_5) = d(0, 1)$. Write $f(p_3) = b(x_1, y_1)$, $f(p_4) = c(x_2, y_2)$ (Fig. 7). We have

$$2A(Oad) = 1, \quad 2A(Oab) = y_1, \quad 2A(Ocd) = x_2, \tag{3}$$

$$\left.\begin{aligned} 2A(abc) &= |\overrightarrow{ab} \times \overrightarrow{ac}| = (x_1y_2 - y_1x_2) + (y_1 - y_2) \text{ and} \\ 2A(bcd) &= |\overrightarrow{db} \times \overrightarrow{dc}| = (x_1y_2 - y_1x_2) + (x_2 - x_1). \end{aligned}\right\} \tag{4}$$

Since $A(Oab) \leq A(Oad)$ and $A(Ocd) \leq A(Oad)$ by (2), it follows from (3) that

$$0 < y_1 \leq 1 \quad \text{and} \quad 0 < x_2 \leq 1. \tag{5}$$

Furthermore, if there exists a triangle $T \in \{Oab, abc, bcd, Ocd\}$ having area at most $\frac{1}{4}A(K)$, then applying Theorem 2 to the quadrilateral $K - T$, we obtain $t_2(K) \geq \frac{8}{9} \cdot \frac{3}{4} = \frac{2}{3}$, as desired. Therefore we may, in particular, assume that

$$A(Oab) + A(Ocd) > \tfrac{1}{2}A(K) \text{ and} \tag{6}$$

$$A(abc) + A(bcd) > \tfrac{1}{2}A(K). \tag{7}$$

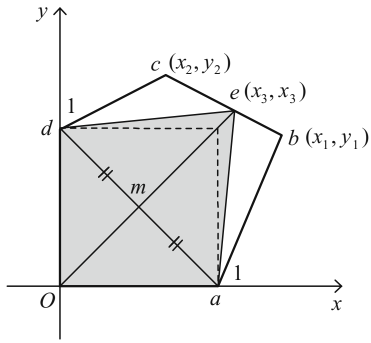


Fig. 7

Since (6) implies $A(Obc) < \frac{1}{2}A(K)$, it follows from (7) that $2[A(abc) + A(bcd)] > 2A(Obc) = |\overrightarrow{Ob} \times \overrightarrow{Oc}| = x_1y_2 - y_1x_2$, and hence

$$(x_1y_2 - y_1x_2) + (x_2 - x_1) + (y_1 - y_2) > 0 \tag{8}$$

by (4). Let m be the midpoint of ad and let $e(x_3, x_3)$ be the intersection point of the straight lines Om and bc. Then $x_3 = \frac{x_1y_2 - y_1x_2}{x_1 - x_2 + y_2 - y_1}$, and hence

$$x_3 > 1 \tag{9}$$

by (8). Thus e is on the side bc, and Oae and Ode are equal area triangles in K. We show $A(Oae) + A(Ode) > \frac{2}{3}A(K)$. Write $A(Oad) = S_1$, $A(ead) = S_2$. Then $S_2 = \frac{em}{Om}S_1 > S_1$ by (9). Furthermore, since $A(abe) + A(cde) \leq \max\{A(abc), A(bcd)\}$, it follows from (2) that $(S_2 >)S_1 \geq A(abe) + A(cde)$. Consequently, $\frac{A(abe)+A(cde)}{A(K)} < \frac{1}{3}$, and hence $\frac{A(Oae)+A(Ode)}{A(K)} > \frac{2}{3}$, as desired.

(ii) Let $\mathcal{P}$ be the set of convex pentagons, and let $\tau = \inf_{P\in\mathcal{P}} t_2(P)$. We first show that $\frac{\tau}{\tau+2} < \frac{5-\sqrt{5}}{10}$. Let $P = r_1r_2r_3r_4r_5$ be a regular pentagon. In view of Propositions 2 and 1 it follows that $\tau \leq t_2(P) \leq \frac{A(r_1r_2r_3r_4)}{A(P)} = \frac{5+\sqrt{5}}{10}$. Thus $\frac{\tau}{\tau+2} = 1 - \frac{2}{\tau+2} \leq \frac{5+\sqrt{5}}{25+\sqrt{5}} < \frac{5-\sqrt{5}}{10}$.

Now consider any convex pentagon $K = p_1p_2p_3p_4p_5$ of area 1. In view of Proposition 3, we may assume $A(p_1p_2p_3) \geq \frac{5-\sqrt{5}}{10}$. Then we can take point q on p_2p_3 such that $A(p_1p_2q) = \frac{\tau}{\tau+2}$. By induction, the pentagon $p_1qp_3p_4p_5$ contains internally disjoint triangles T_1 and T_2 such that $A(T_1) = A(T_2) = \frac{\tau}{2} \times A(p_1qp_3p_4p_5) = \frac{\tau}{\tau+2} = A(p_1p_2q)$. Thus $t_3(K) \geq \frac{3\tau}{\tau+2}$. Since $\tau \geq \frac{2}{3}$ by (i), $t_3(K) \geq 3\left(1 - \frac{2}{\tau+2}\right) \geq \frac{3}{4}$, as desired.

(iii) We may assume that $A(K) = 1$ and $A(p_1p_2p_3) \geq \frac{5-\sqrt{5}}{10}$ (recall Proposition 3). The proof is by induction on n. We first show that $t_4(K) \geq \frac{8}{9}$. By Proposition 4, we can divide K into two convex polygons Q_1 and Q_2 each with at most four sides and $A(Q_1) = A(Q_2) = \frac{1}{2}$. Hence by Theorem 2, we can take internally disjoint triangles $T_1, T_2 \subset Q_1$ and $T_3, T_4 \subset Q_2$ such that $A(T_1) = \cdots = A(T_4) = \frac{4}{9} \times \frac{1}{2} = \frac{2}{9}$. Thus $t_4(K) \geq \frac{8}{9}$. Next suppose that $t_n(K) \geq \frac{2n}{2n+1}$ for some $n \geq 4$. Take point q on p_2p_3 such that $A(p_1p_2q) = \frac{2}{2(n+1)+1} \left(< \frac{5-\sqrt{5}}{10}\right)$. By our induction hypothesis, the pentagon $p_1qp_3p_4p_5$ contains n internally disjoint triangles $T_1, \cdots, T_n$ such that $A(T_1) = \cdots = A(T_n) = \frac{2}{2n+1} \times A(p_1qp_3p_4p_5) = \frac{2}{2(n+1)+1} = A(p_1p_2q)$. Consequently, $t_{n+1}(K) \geq \frac{2(n+1)}{2(n+1)+1}$, as desired. □

For a positive integer n and a regular hexagon K, we have by Theorem D that $t_{6n}(K) = 1$. We show here that:

Theorem 4. *Let $n \geq 2$ be an integer and let K be a convex polygon with at most six sides. Then $t_{3n}(K) \geq \frac{4n}{4n+1}$.*

Proof. We may assume that $A(K) = 6$. We first show that K can be divided into two polygons K_1 and K_2 such that K_1 has at most four sides, $A(K_1) = 2$, K_2 has at most five sides and $A(K_2) = 4$. Let $K = p_1p_2 \cdots p_6$. In the case where K has $k < 6$ sides, take $6-k$ points on one of its edges, and think of them as $6-k$ artificial vertices of K which can now be considered as a convex hexagon. Write $T_1 = p_1p_2p_3$, $T_2 = p_3p_4p_5$, $T_3 = p_5p_6p_1$. We may assume that $A(T_1) < 2$. First consider the case where $A(T_3) < 2$. In this case, we may assume further that $A(p_1p_2p_3p_4) \geq 3$ by symmetry. Then there exists a point $q \in p_3p_4$ such that $A(p_1p_2p_3q) = 2$ and $A(p_1qp_4p_5p_6) = 4$, as desired. Thus consider the case where $A(T_3) \geq 2$. Since $A(T_1) < 2$ and $A(p_1p_2p_3p_5p_6) > A(T_3) \geq 2$, either there exists a point $q \in p_6p_1$ such that $A(p_1p_2p_3q) = 2$, or there exists a point $q \in p_5p_6$ such that $A(p_1p_2p_3qp_6) = 2$. In the former case, $K_1 = p_1p_2p_3q$ and $K_2 = p_3p_4p_5p_6q$ have the desired properties. In the latter case, since $A(p_1p_2qp_6) < A(p_1p_2p_3qp_6) = 2$ and $A(p_1p_2p_5p_6) > A(T_3) \geq 2$, there exists a point $r \in qp_5$ such that $A(p_1p_2rp_6) = 2$ and $A(p_2p_3p_4p_5r) = 4$, as desired.

Now since $t_n(K_1) \geq \frac{4n}{4n+1}$ by Theorem 2 (ii) and $t_{2n}(K_2) \geq \frac{4n}{4n+1}$ by Theorem 3 (iii), we obtain $t_{3n}(K) \geq \frac{4n}{4n+1}$, as desired. □

4 Equal Area Polygons in a Convex Body

Let K be a convex body in $\boldsymbol{E}^2$. Combining Theorem B and Proposition 4, we obtain several results. For example, for any integer $n \geq 3$,

$$\frac{2n-3}{2\pi} \sin \frac{2\pi}{2n-3} \leq s(K; 1, 2n-3) \leq s(K; 2, n). \tag{10}$$

Similarly, for $m = 2^l$, $l = 0, 1, 2, \cdots$, we have $s(K; m, 4) \geq s(K; \frac{m}{2}, 5) \geq \cdots \geq s(K; 1, m+3) \geq \frac{m+3}{2\pi} \sin \frac{2\pi}{m+3}$, and hence

$$s(K; 2m, 3) \geq \frac{8}{9} s(K; m, 4) \geq \frac{4(m+3)}{9\pi} \sin \frac{2\pi}{m+3} \tag{11}$$

by Theorem 2. On the other hand, it follows from Proposition 2 that

$$s(K; 2, n) \leq s(K; 1, 2n-2). \tag{12}$$

We henceforth focus on $s(K; 2, n)$. By (10) and (12),

$$s(K; 1, 2n-3) \leq s(K; 2, n) \leq s(K; 1, 2n-2) \text{ for } n \geq 3, \tag{13}$$

and by (11) and (12),

$$\tfrac{8}{9} s(K; 1, 4) \leq s(K; 2, 3) \leq s(K; 1, 4). \tag{14}$$

We believe that the following is true:

Conjecture 1. Let K be a convex body in $\boldsymbol{E}^2$. Then $s(K; 2, n) \geq \frac{n-1}{\pi} \sin \frac{\pi}{n-1}$ with equality if and only if K is an ellipse.

Remark 3. We can verify that the equality of this conjecture holds if K is an ellipse in the following way: Let E be an ellipse. Since a circular disk D contains a regular $2(n-1)$-gon R with $A(R) = \frac{n-1}{\pi} \sin \frac{\pi}{n-1} A(D)$, E contains a centrally symmetric $2(n-1)$-gon P with $A(P) = \frac{n-1}{\pi} \sin \frac{\pi}{n-1} A(E)$, which can be divided into two internally disjoint equal area n-gons. Thus $s(E; 2, n) \geq \frac{n-1}{\pi} \sin \frac{\pi}{n-1}$. Furthermore, we have $s(E; 2, n) \leq s(E; 1, 2n-2) = \frac{n-1}{\pi} \sin \frac{\pi}{n-1}$ by (12) and Theorem B. Consequently, $s(E; 2, n) = \frac{n-1}{\pi} \sin \frac{\pi}{n-1}$ holds for any ellipse E.

In this section, we settle Conjecture 1 affirmatively for some special cases.

Theorem 5. *Let K be a centrally symmetric convex body in $\boldsymbol{E}^2$. Then $s(K; 2, n) \geq \frac{n-1}{\pi} \sin \frac{\pi}{n-1}$.*

To prove Theorem 5, it suffices to show the following:

$$\left.\begin{array}{l}\text{Let } K \text{ be a centrally symmetric convex body in } \boldsymbol{E}^2\text{. Then there exists} \\ \text{a polygon } P \subseteq K \text{ with } \frac{A(P)}{A(K)} \geq \frac{n-1}{\pi} \sin \frac{\pi}{n-1} \text{ such that } P \text{ has at most} \\ 2n-2 \text{ sides and } P \text{ is centrally symmetric with respect to the center} \\ \text{of } K. \end{array}\right\} \tag{15}$$

Observe that P can then be divided into two internally disjoint equal area polygons with at most n sides. We show (15) in a generalized form stated in the following Theorem 6.

Let n be a positive integer. For a subset S of $\boldsymbol{E}^n$ having a finite volume, let $V(S)$ denote the volume of S. For a centrally symmetric convex body K in $\boldsymbol{E}^n$, denote by $\mathcal{Q}_m(K)$ the set of polytopes P contained in K such that P is centrally symmetric with respect to the center of K and P has at most $2m$ vertices. Let

$$\sigma(K; m) = \sup_{P \in \mathcal{Q}_m(K)} \frac{V(P)}{V(K)}.$$

Theorem 6. *Let m and n be integers with $m \geq n \geq 2$. Let K be a centrally symmetric convex body in $\boldsymbol{E}^n$ and let S be a hyper-sphere in $\boldsymbol{E}^n$. Then $\sigma(K; m) \geq \sigma(S; m)$.*

Proof. Our proof is a modification of the proof of the n-dimensional theorem of Theorem B by Macbeath [8], where *Steiner symmetrization* is applied. We give only a sketch of our proof.

We may assume that K is centrally symmetric with respect to the origin O of $\boldsymbol{E}^n$. Let π be a hyper-plane in $\boldsymbol{E}^n$ containing the origin O. Denote each point a in $\boldsymbol{E}^n$ by (x,t), where $x = x(a)$ is the foot of the perpendicular from a to π and $t = t(a)$ is the oriented perpendicular distance from x to a. For a convex body B, let B' be the projection of B on π. For $x \in B'$, define the two functions $B^+(x)$ and $B^-(x)$ by $B^+(x) = \sup_{(x,t)\in B} t$ and $B^-(x) = \inf_{(x,t)\in B} t$. Then $B = \{\,(x,\,t)\,|\,x \in B',\ B^-(x) \le t \le B^+(x)\,\}$.

Let $K^* = \{\,(x,\,t)\,|\,x \in K',\ |t| \le \frac{1}{2}(K^+(x) - K^-(x))\,\}$. Then K^* is symmetric with respect to π, centrally symmetric with respect to O, and $V(K^*) = \int_{K'}(K^+(x) - K^-(x))dx = V(K)$. By the central symmetry of K with respect to O,

$$-x \in K',\ K^+(-x) = -K^-(x) \text{ and } K^-(-x) = -K^+(x) \text{ for any } x \in K'. \quad (16)$$

Lemma 1. $\sigma(K^*; m) \le \sigma(K; m)$

Proof. Let P be a polytope in $\mathcal{Q}_m(K^*)$. It suffices to show that there is a polytope $P_0 \in \mathcal{Q}_m(K)$ such that $V(P_0) \ge V(P)$. Let $2k\,(\le 2m)$ be the number of vertices of P and let $(x_i,\,t_i), 1 \le i \le 2k$, be the vertices of P. We label the indices so that for each $1 \le i \le k$, $(x_i,\,t_i)$ and $(x_{k+i},\,t_{k+i})$ are symmetric with respect to O (so $(x_{k+i},\,t_{k+i}) = (-x_i,\,-t_i)$). Let Q be the convex hull of the points $(x_i,\,t_i + \frac{1}{2}(K^+(x_i) + K^-(x_i)))$, $1 \le i \le 2k$, and let R be the convex hull of the points $(x_i,\,-t_i + \frac{1}{2}(K^+(x_i) + K^-(x_i)))$, $1 \le i \le 2k$. Since $|t_i| \le \frac{1}{2}(K^+(x_i) - K^-(x_i))$, each vertex of Q and R is contained in K, and hence Q, $R \subseteq K$. Also, since for each $1 \le i \le k$,

$$\begin{aligned}
&\tfrac{1}{2}(x_i + x_{k+i}) = \tfrac{1}{2}(x_i + (-x_i)) = 0 \qquad \text{and}\\
&\tfrac{1}{2}\left[t_i + \tfrac{1}{2}(K^+(x_i) + K^-(x_i)) + t_{k+i} + \tfrac{1}{2}(K^+(x_{k+i}) + K^-(x_{k+i}))\right]\\
&\quad = \tfrac{1}{2}\left[t_i + \tfrac{1}{2}(K^+(x_i) + K^-(x_i)) + (-t_i) + \tfrac{1}{2}(K^+(-x_i) + K^-(-x_i))\right]\\
&\quad = \tfrac{1}{2}\left[\tfrac{1}{2}(K^+(x_i) + K^-(-x_i)) + \tfrac{1}{2}(K^+(-x_i) + K^-(x_i))\right]\\
&\quad = 0
\end{aligned}$$

by (16), Q is centrally symmetric with respect to O. Similarly, we see that R is centrally symmetric with respect to O. Furthermore, since

$$\begin{aligned}
Q^-(x_i) &\le t_i + \tfrac{1}{2}(K^+(x_i) + K^-(x_i)) \le Q^+(x_i) \quad \text{and}\\
R^-(x_i) &\le -t_i + \tfrac{1}{2}(K^+(x_i) + K^-(x_i)) \le R^+(x_i),
\end{aligned}$$

we have that $\frac{1}{2}(Q^-(x_i) - R^+(x_i)) \le t_i \le \frac{1}{2}(Q^+(x_i) - R^-(x_i))$, and hence each point $(x_i,\,t_i)$, $1 \le i \le 2k$, lies in the convex set

$$T = \{\,(x,\,t)\,|\,x \in P',\ \tfrac{1}{2}(Q^-(x) - R^+(x)) \le t \le \tfrac{1}{2}(Q^+(x) - R^-(x))\,\}.$$

Since P is the convex hull of the points $(x_i,\, t_i)$, $1 \le i \le 2k$,

$$V(P) \le V(T) = \tfrac{1}{2}\int_{P'}(Q^+(x) - Q^-(x) + R^+(x) - R^-(x))\,dx$$
$$= \tfrac{1}{2}(V(Q) + V(R)).$$

Thus at least one of $V(Q) \ge V(P)$ or $V(R) \ge V(P)$ holds. Consequently, Q or R is a polytope with desired properties. □

Now we return to the proof of Theorem 6. The rest of our argument follows exactly as the proof in [8]: we can verify that $\sigma(K;m)$ is a continuous function of K. Let $\pi_1, \pi_2, \cdots, \pi_n$ be n hyper-planes such that for each pair $i \neq j$ π_i and π_j form an angle which is an irrational multiple of π. Consider the sequence of bodies $K = K_1, K_2, \cdots, K_n, \cdots$, where K_{i+1} arises from K_i by symmetrizing it with respect to π_ν where ν is the least positive residue of $i \pmod n$. This sequence converges to a hyper-sphere S (see [3]), and hence $\sigma(K;m) \ge \sigma(S;m)$. □

Let K be a convex body in $\boldsymbol{E}^2$ and let l denote the perimeter of K. Then

$$\textit{The Isoperimetric Inequality:}\quad l^2 \ge 4\pi A(K) \tag{17}$$

with equality if and only if K is a circular disk; and, if K is a figure with constant width w, we also have

$$\textit{Barbier's Theorem:}\quad l = \pi w \tag{18}$$

(see, for example, [4]). Finally we show that Conjecture 1 is true for $n = 3$ when K is a figure with constant width:

Theorem 7. *Let K be a figure with constant width in $\boldsymbol{E}^2$. Then $t_2(K) \ge \frac{2}{\pi}$ with equality if and only if K is a circular disk.*

Proof. Let w and l denote the width and perimeter of K, respectively. For each $\theta \in [0, 2\pi)$, let $\boldsymbol{u}_\theta$ denote the vector $(\cos\theta, \sin\theta)$, let $a = a_\theta$ and $b = b_\theta$ denote the contact points of K and each of two straight lines parallel to $\boldsymbol{u}_\theta$, and let $m = m_\theta$ denote the midpoint of the segment ab (Fig. 8 (a)). Let $c = c_\theta$ and $d = d_\theta$ be the intersection points of the perimeter of K and the straight line passing through m and parallel to $\boldsymbol{u}_\theta$. Then we have $A(acd) = A(bcd)$. Take c' on the line tangent to the perimeter of K at c such that $\det\left[\overrightarrow{cc'}\ \boldsymbol{u}_\theta\right] > 0$, where $\left[\overrightarrow{cc'}\ \boldsymbol{u}_\theta\right]$ stands for a matrix having $\overrightarrow{cc'}$ and $\boldsymbol{u}_\theta$ as their column vectors. We further take d' on the tangent line of the perimeter of K at d such that $\det\left[\overrightarrow{dd'}\ \boldsymbol{u}_\theta\right] < 0$. Write $\alpha_1 = \alpha_1(\theta) = \angle mcc'$ and $\alpha_2 = \alpha_2(\theta) = \angle mdd'$.

Since $\alpha_1(\theta+\pi) - \alpha_2(\theta+\pi) = -(\alpha_1(\theta) - \alpha_2(\theta))$ (Fig. 8 (a),(b)), it follows from the Intermediate Value Theorem that there exists $\theta \in [0, \pi]$ such that $\alpha_1(\theta) - \alpha_2(\theta) = 0$ i.e. $cc' \parallel dd'$. For this θ, we have $cd \ge w$, and hence

$$A(acd) + A(bcd) = \tfrac{1}{2}cd \cdot w \ \ge\ \tfrac{1}{2}w^2 = \tfrac{1}{2}\left(\tfrac{l}{\pi}\right)^2 \ \ge\ \tfrac{2}{\pi}\cdot A(K)$$

by (17) and (18). Furthermore, if $t_2(K) = \frac{\pi}{2}$ holds, then we must have $l^2 = 4\pi A(K)$, i.e., K is a circular disk; and for a circular disk K, we have $t_2(K) = \frac{\pi}{2}$ (recall Remark 3). □

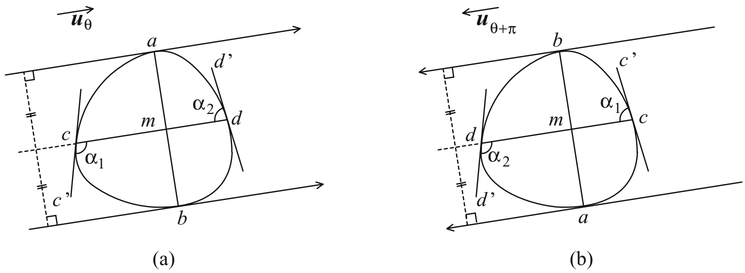


Fig. 8

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Maximum Order of Planar Digraphs

Rinovia Simanjuntak and Mirka Miller

School of Electrical Engineering and Computer Science,
The University of Newcastle, NSW 2308, Australia
{rino, mirka}@cs.newcastle.edu.au

Abstract. We consider the *degree/diameter* problem for directed planar graphs. We show that planar digraphs with diameter 2 and maximum out-degree and in-degree d, $d \geq 41$, cannot have more than $2d$ vertices. We show that $2d$ is the best possible upper bound by constructing planar digraphs of diameter 2 having exactly $2d$ vertices.

Furthermore, we give upper and lower bounds for the largest possible order of planar digraphs with diameter greater than 2.

1 Introduction

One of the famous and difficult graph-theoretical problems over the past four decades is the *degree/diameter problem* which is to determine, for each d and k, the largest order $n_{d,k}$ of a graph of maximum degree d and diameter at most k. Trivially, $n_{d,k}$ cannot be larger than the so-called Moore bound, $M_{d,k} = 1 + d + d(d-1) + d(d-1)^2 + \ldots + d(d-1)^{k-1}$. However, it is not trivial to show that the Moore bound is attained only if $k = 1$ and $d \geq 1$ or if $k = 2$ and $d = 2,\ 3,\ 7$, and possibly 57, or else if $k \geq 3$ and $d = 2$ [11, 1, 5]. For the remaining values of $d \geq 3$ and $k \geq 2$, results of Bannai and Ito [2] show that $n_{d,k} \leq M_{d,k} - 2$. A further improvement for cubic graphs was obtained by Jørgensen [12] who showed that $n_{3,k} \leq M_{3,k} - 4$ for $k \geq 4$.

Based on these results, it seems reasonable to study extremal graphs subject to some further restrictions. Since in certain applications crossings among connections are not allowed, planarity of the graphs becomes a natural requirement. Thus we shall consider the degree/diameter problem for planar graphs. In [10], Hell and Seyffarth proved that a planar graph of diameter 2 and maximum degree $d \geq 8$ has at most $b = \lfloor \frac{3}{2}d \rfloor + 1$ vertices. They proved the result by considering several subgraphs in a planar graph of order more than b and showing that such a graph is forced to have diameter larger than 2. Additionally, as shown in [18], the bound is the best possible since for every $d \geq 8$ there exists a planar triangulation of diameter 2 and maximum degree d containing precisely $\lfloor \frac{3}{2}d \rfloor + 1$ vertices (for illustration, see Fig. 1).

For $k \geq 3$, Fellows *et al.* [7] proved that every planar graph of diameter k and maximum degree $d \geq 4$ has at most $(6k+3)(2d^{\lfloor \frac{k}{2} \rfloor} + 1)$ vertices. They also gave an improved bound for diameter 3, namely, $8d + 12$. In general, unlike in the case of diameter 2, the problem of whether the bound can be achieved or

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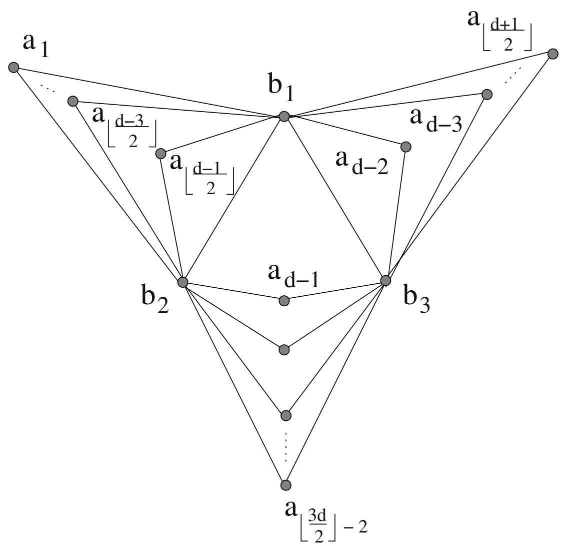


Fig. 1. Planar graphs of diameter 2, maximum degree $d \geq 8$ and maximum order

not is still open. None of the constructions which yield the largest known planar graphs (see [6, 8, 9, 17]) has so far produced graphs achieving the upper bound. More specifically, we have the following

Open Problem 1. For $k = 3$ and $d \geq 4$, do there exist planar graphs of diameter k and maximum degree d with order $8d + 12$?

Open Problem 2. For $k \geq 4$ and $d \geq 4$, do there exist planar graphs of diameter k and maximum degree d with order $(6k + 3)(2d^{\lfloor \frac{k}{2} \rfloor} + 1)$?

The directed version of the degree/diameter problem has also been considered during the last thirty years. It is well known that the trivial bound for the order of a digraph of maximum out-degree d and diameter at most k, $n^*_{d,k}$, is $1 + d + d^2 + \ldots + d^k$; the bound is also called the Moore bound but to avoid confusion with the undirected case, we shall denote the directed Moore bound by $M^*_{d,k}$. In [16], Plesník and Znám (see also [4]) proved that the Moore bound, $M^*_{d,k}$, is attained only when $d = 1$ or $k = 1$. For general $d \geq 2$, we know that if $k = 2$, there exist digraphs of order $M^*_{d,k} - 1$, which are the line digraphs of complete digraphs for $d \geq 3$ plus two other digraphs for $d = 2$. However, for the remaining values of $d \geq 3$ and $k \geq 3$, the existence of digraphs of order $M^*_{d,k} - 1$ is completely open. Some improvements for $d = 2$ was obtained by Miller and Fris [14] and Miller and Širáň [15] who showed that $n^*_{2,k} \leq M^*_{2,k} - 3$, and for $d = 3$ a recent result [3] showed that $n^*_{3,k} \leq M^*_{3,k} - 2$.

Our paper [19] considered for the first time the degree/diameter problem for planar digraphs:

> For given d and k, find a directed planar graph of maximum out-degree and in-degree d and diameter at most k with maximum number of vertices.

Morover, in [19] we gave a sketch of a proof of the following theorem

Theorem 1. Let $d \geq 41$. If D is a digraph of diameter 2, maximum out-degree d and maximum in-degree d, embeddable in the plane, then $|V(D)| \leq 2d$ and this bound is the best possible.

In this paper, we shall present the proof in detail. We shall also give upper and lower bounds for the number of vertices of planar digraphs with diameter larger than 2.

All digraphs considered in this paper are finite and connected. Unless otherwise stated, from now on, we consider D to be a planar digraph; and, as usual, we denote by $V(D)$ and $E(D)$ the vertex and the arc sets of D, respectively. For u and v vertices in D, (u, v) denotes an arc from u to v and $dist(u, v)$ stands for the distance between u and v. For arbitrary $u \in V(D)$ we define $N_i^+(u) = \{v \in V(D) : dist(u, v) = i\}$ and, correspondingly, $N_i^-(u) = \{v \in V(D) : dist(v, u) = i\}$. The out-degree of u is denoted by $deg_D^+(u)$ or simply $deg^+(u)$ and, similarly, $deg_D^-(u)$ or $deg^-(u)$ denotes the in-degree of u in D.

2 Largest Planar Digraphs of Diameter 2

In this section we restrict our attention to the degree/diameter problem for planar digraphs of diameter two and our aim is to prove Theorem 1. Before proving the Theorem, we shall introduce two families of planar digraphs which, as we shall observe later, are of principal importance in our investigation. By a *dipole*, $D_t = D_t(x, y)$, we mean the digraph with vertex set $V(D_t) = \{x, y, x_1, \ldots x_t\}$ and edge set $E(D_t) = \{(x, x_i), (x_i, y); i = 1, \ldots, t\}$. We start with a simple observation.

Lemma 1. Suppose $D_t(x, y) \subset D$ and $v \in V(D \backslash D_t)$ has distance not more than 2 from each x_i. If $t \geq 5$ then $(v, x) \in E(D)$ or $(v, y) \in E(D)$.

Now we shall consider a digraph D_t^* constructed from the dipole D_t where directed paths of length 2 are replaced by directed paths of length 3. Let $D_t^* = D_t^*(x, y)$ be the digraph with vertex set $V(D_t^*) = \{x, y, x_1, y_1, \ldots, x_t, y_t\}$ and edge set $E(D_t^*) = \{(x, x_i), (x_i, y_i), (y_i, y); i = 1, \ldots, t\}$; we shall refer to this digraph as a *twice-dipole*. It is easy to see that the following holds.

Lemma 2. Suppose $D_t^*(x, y) \subset D$ and $(x, y_i) \notin E(D)$ for each i. Let $v \in V(D \backslash D_t^*)$ have distance not more than 2 from each x_i and y_i. If $t \geq 5$ then $(v, x) \in E(D)$ and $(v, y) \in E(D)$.

2.1 Proof of Theorem 1

Assume the contrary and let D be the digraph with the following properties:

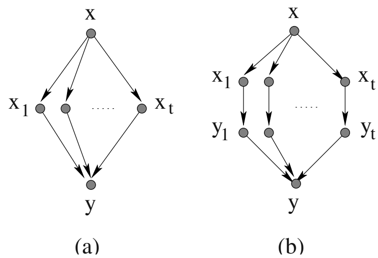


Fig. 2. (a) The dipole D_t. (b) The twice-dipole D_t^*

(i) D has diameter 2,
(ii) the maximum out-degree and in-degree of D are each at least d,
(iii) D is embeddable in the plane, and
(iv) $n = |V(D)| \geq 2d + 1$.

Choose any vertex u in D. Since the maximum out-degree of u is at least d and the order of D is at least $2d + 1$, then there must be at least d vertices at distance 2 from u. Our aim is to derive an upper bound on the out-degree of u so that we obtain a contradiction with **(ii)**, through the following steps.

Claim 1. If $dist(u, v) = 2$ then $|N_1^+(u) \cap N_1^-(v)| \leq 4$.

Proof of Claim 1. We use contradiction to prove this claim, by investigating two cases, (a) $5 \leq |N_1^+(u) \cap N_1^-(v)| \leq d-2$ and (b) $d-1 \leq |N_1^+(u) \cap N_1^-(v)| \leq d$. In the first case, we apply Lemma 1 to the dipole $D_{|N_1^+(u) \cap N_1^-(v)|}(u, v)$, so that every vertex $x \notin N_1^+(u) \cap N_1^-(v) \cup \{u, v\}$ is connected to u. Since $|D \backslash (N_1^+(u) \cap N_1^-(v) \cup \{u, v\})| \geq 2 + (d - 1) = d + 1$ then $deg^-(u) \geq d + 1$, a contradiction.

In the second case, i.e., $d-1 \leq |N_1^+(u) \cap N_1^-(v)| \leq d$, we first apply Lemma 1 to $D_{|N_1^+(u) \cap N_1^-(v)|}(u, v)$. Thus $(x, u) \in E(D)$ for each $x \notin N_1^+(u) \cap N_1^-(v) \cup \{u, v\}$. Since the diameter of D is 2 then u has to reach all vertices in $N_2^+(u) \backslash \{v\}$ using a directed path of length 2; and for this purpose, there are three possible vertices that could be utilised. These are the two "outside" vertices in $N_1^+(u) \cap N_1^-(v)$ (i.e., the two vertices that share only one common face with the other vertices in $N_1^+(u) \cap N_1^-(v)$) and, if $|N_1^+(u) \cap N_1^-(v)| = d - 1$, the one vertex in $N_1^+(u) \backslash N_1^+(u) \cap N_1^-(v)$. Without lost of generality, one of these three vertices, say w, will be connected to at least $\lfloor \frac{d-1}{3} \rfloor$ vertices in $N_2^+(u) \backslash \{v\}$. Say that w is connected to m vertices in $N_2^+(u) \backslash \{v\}$, with $\lfloor \frac{d-1}{3} \rfloor \leq m \leq d - 1$. Now, consider the dipole $D_q(w, u)$. By Lemma 1, every vertex not in the dipole will be connected to w. Therefore, $deg^-(w) \geq 1 + (d - 1) + 1 + (d - 1 - m) \geq 2d - (d - 1) \geq d + 1$, a contradiction, and so we obtain the desired result, that is, $|N_1^+(u) \cap N_1^-(v)| \leq 4$. □

Choose a vertex $v \in N_2^+(u)$ so that the cardinality of the set $\mathcal{A} = N_1^+(u) \cap N_1^-(v)$ is the largest possible. Next we denote $\mathcal{A}' = N_1^+(u) \backslash \mathcal{A}$ and then consider all possible directed paths of length 2 from a vertex $w \in \mathcal{A}'$ to v. We shall deal separately with directed paths $P = (w, x, v)$ where $x \in \mathcal{A}$ or where $x \in N_2^+(u) \cap N_1^-(v)$ and $deg^-_{N_1^+(u)\setminus\mathcal{A}}(x) > 1$ or where $x \in N_2^+(u) \cap N_1^-(v)$ and $deg^-_{N_1^+(u)\setminus\mathcal{A}}(x) = 1$.

We can now define the following sets (see Fig. 3)

$$\mathcal{B} = \{w \in \mathcal{A}' : \exists P = (w, x, v), x \in \mathcal{A}\},$$

$$\mathcal{C} = \{w \in \mathcal{A}' : \exists P = (w, x, v), x \in N_2^+(u) \cap N_1^-(v) \ \&\ deg^-_{N_1^+(u)\setminus\mathcal{A}}(x) > 1\}, \text{ and}$$

$$\mathcal{D} = \{w \in \mathcal{A}' : \exists P = (w, x, v), x \in N_2^+(u) \cap N_1^-(v) \ \&\ deg^-_{N_1^+(u)\setminus\mathcal{A}}(x) = 1\}.$$

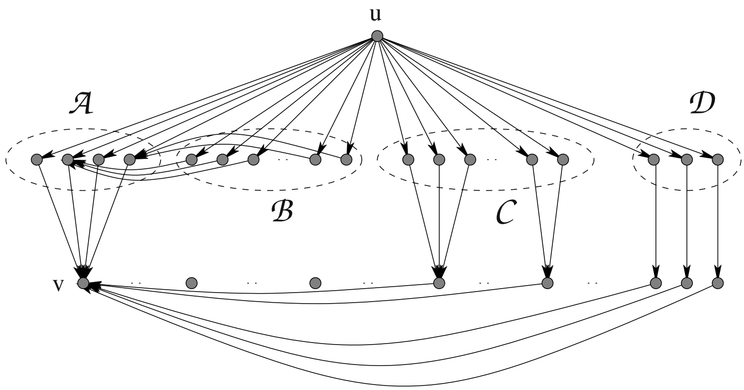


Fig. 3. Illustration of the sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$

Claim 2. There can be at most 16 vertices in $\mathcal{B}$.

Proof of Claim 2. Denote by $a_i, i = 1, \ldots, |\mathcal{A}|$, all vertices in $\mathcal{A}$. Define the sets $\mathcal{B}_1, \ldots, \mathcal{B}_{|\mathcal{A}|}$ in such a way that $x \in \mathcal{B}_i$ if $(x, a_i) \in E(D)$. For every $i = 1, \ldots |\mathcal{A}|$, consider the dipole $D_{|\mathcal{B}_i|}(u, a_i)$. Suppose that $|\mathcal{B}_i| \geq 5$, then, by applying Lemma 1, every vertex that is not in $\mathcal{B}_i \cup \{u, a_i\}$ will be connected to u. Thus, $deg^-(u) = d - |\mathcal{B}_i| + N_2^+(u) \geq |\mathcal{A}| + d > d$, a contradiction. It remains to consider $|\mathcal{B}_i| \leq 4$ for each i. Since $\mathcal{A} \leq 4$ then $\mathcal{B} \leq \mathcal{B}_1 + \ldots + \mathcal{B}_4 \leq 4 + 4 + 4 + 4 = 16$. □

Claim 3. There can be at most 16 vertices in $\mathcal{C}$.

Proof of Claim 3. For $j = 1, \ldots, n - (d + 2)$, denote by v_j all vertices in $N_2^+(u)\setminus\{v\}$ with $deg^-_{N_1^+(u)\setminus\mathcal{A}}(v_j) > 1\}$ and by $\mathcal{C}_j$ a subset of $N_1^+(u)\setminus\{v\}$ with the property that $x \in \mathcal{C}_j$ if (x, v_j) and $(v_j, v) \in E(D)$. By the definition of $\mathcal{A}$, $|\mathcal{C}_j| \leq |\mathcal{A}| \leq 4$, for all $j = 1, \ldots, n - (d + 2)$.

To prove that there exist at most 4 $\mathcal{C}_j$'s we shall define the set $\mathcal{T} = \{v_j \in N_2^+(u)\setminus\{v\}|\ deg^-_{N_1^+(u)\setminus\mathcal{A}}(v_j) > 1 \text{ and } (v_j, v) \in D\}$. Now consider consider the twice-dipole $D^*_{|\mathcal{T}|}(u,v)$. Suppose that $|\mathcal{T}| \geq 5$, by Lemma 2, all vertices $\notin D^*_{|\mathcal{T}|}(u,v)$ are connected to v.

Since all vertices in $\mathcal{A}$ and all vertices in $\mathcal{T}$ are connected to v then there exist at least $|\mathcal{A}| + |\mathcal{T}|$ arcs incident to v. Thus

$$deg^-(v) \geq |\mathcal{A}| + |\mathcal{T}| + (d - |\mathcal{A}| - |\mathcal{T}|) + (d - 1 - |\mathcal{T}|) = 2d - 1 - |\mathcal{T}|.$$

On the other hand, $|\mathcal{A}|, |\mathcal{B}|, |\mathcal{C}| \geq 1$ and so $|\mathcal{T}| \leq d - |\mathcal{A}| - |\mathcal{B}| - |\mathcal{C}| \leq d - 3$. Hence,

$$deg^-(v) \geq 2d - 1 - (d - 3) = d + 2,$$

a contradiction which gives $|\mathcal{T}| \leq 4$ and so there exist at most 4 $\mathcal{C}_j$'s.

Thus $\mathcal{C} \leq \mathcal{C}_1 + \ldots + \mathcal{C}_4 \leq 4 + 4 + 4 + 4 = 16$. □

Claim 4. There can be at most 4 vertices in $\mathcal{D}$.

Proof of Claim 4. Consider the twice-dipole $D^*_{|\mathcal{D}|}(u,v)$. Suppose that $|\mathcal{D}| \geq 5$, by Lemma 2, there exists an arc from every vertex $x \notin D^*_{|\mathcal{D}|}(u,v)$ to v. Since there are already $|\mathcal{A}|+|\mathcal{D}|$ arcs connected to v (from all vertices in $\mathcal{A}$ and from all vertices in $\mathcal{D}$), it follows that $deg^-(v) \geq |\mathcal{A}|+|\mathcal{D}|+(d-|\mathcal{A}|-|\mathcal{D}|)+(d-1-|\mathcal{D}|) = 2d - 1 - |\mathcal{D}|$.

On the other hand, it is obvious that $|\mathcal{A}|, |\mathcal{B}|, |\mathcal{C}| \geq 1$ and so $|\mathcal{D}| \leq d - |\mathcal{A}| - |\mathcal{B}| - |\mathcal{C}| \leq d - 3$. Hence, $deg^-(v) \geq 2d - 1 - (d - 3) = d + 2$, a contradiction which gives $|\mathcal{D}| \leq 4$. □

The number of vertices in each of the sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and $\mathcal{D}$ is independent of the cardinalities of the other three sets. Thus

$$\begin{aligned} N_1^+(u) &\leq |\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| + |\mathcal{D}| \\ &\leq 4 + 16 + 16 + 4 = 40, \end{aligned}$$

which gives a contradiction, and so completes the proof of the first part of Theorem 1.

That the bound is the best possible follows from the following example of a planar digraph of maximum degree $d \geq 41$ and diameter 2, having precisely $2d$ vertices (see Fig. 4).

2.2 Conjectures and Open Problems

Although we were able to prove Theorem 1 only for $d \geq 41$, we believe that the Theorem also holds for smaller values of d.

Conjecture 1. Theorem 1 holds for all $d \geq 6$.

For the rest of the cases, i.e., $1 \leq d \leq 5$, we know that one of the Moore digraphs, C_3, is a planar digraph of $d = 1$ and $k = 2$; and for $d = 2$, the three non-isomorphic digraphs of order $M^*_{2,2} - 1$ are also planar (see Fig. 5).

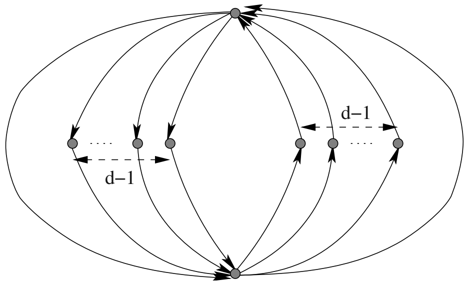
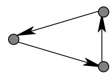
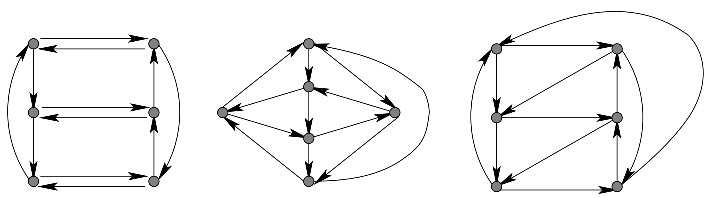


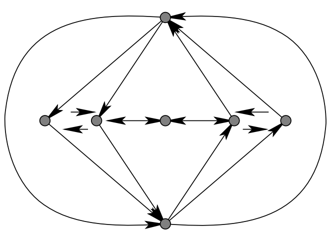
Fig. 4. Planar digraph of maximum out-degree degree $d \geq 41$ and diameter 2, having precisely $2d$ vertices



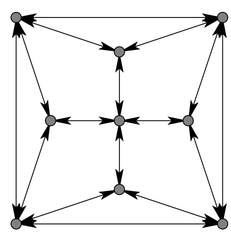
The directed cycle C_3



The three non−isomorphic digraphs of order $M^{*}_{2,\overline{2}}\,1$



Planar digraph of d=3 and n=7



Planar digraph of d=4 and n=9

Fig. 5. Large planar digraphs of maximum out-degree $d = 1, 2, 3, 4$ and diameter 2

For $d = 3$, one vertex can be added in the graph of Fig. 4 without increasing the maximum out-degree and diameter of the graph, thus we have a graph of maximum out-degree 3 and order 7. From a construction for undirected planar graph [6] of maximum degree 4, we can obtain a planar digraph of $d = 4$ on 9 vertices (see Fig. 5).

Open Problem 3. Find the largest planar digraphs of diameter 2 and $d = 3, 4, 5$.

3 Bounds on the Order of Planar Digraphs of Diameter Greater Than 2

For diameter $k \geq 2$, an upper bound for the order of a planar digraph can be found by using a similar argument to the one used by Fellows, Hell, and Seyffarth [7], which combined a simple counting argument with a planar Separator Lemma of Lipton and Tarjan [13]. We recall the lemma of Lipton and Tarjan in the following.

Lemma 3. Let G be a planar graph of order n. Assume that G contains a spanning tree of radius r, rooted at a vertex v. Then the vertex set of G can be partitioned into three subsets A, B, C such that $|A|, |B| \leq \frac{2}{3}n$, $|C| \leq 2r + 1$, $v \in C$, and there is no edge between A and B.

Now we are ready to prove the following theorem.

Theorem 2. Let D be a planar digraph of maximum out-degree d and diameter at most k. Then

$$|V(D)| \leq (6k + 3)\frac{(d^{\lfloor k/2 \rfloor + 1} - 1)}{d - 1}.$$

Proof. Considering the underlying graph G of the digraph D, it is clear that G has diameter at most k, and so contains a spanning tree of radius at most k, rooted at a vertex v. Due to Lemma 3 with $r = k$, there is a partition of $V(G)$, which is of course exactly the same as the partition of $V(D)$, into subsets A, B, C such that $|A|, |B| \leq \frac{2}{3}n$, $|C| \leq 2r + 1$; with no edge between A and B, and so no arcs between A and B.

Assume that there exist vertices $a \in A$ and $b \in B$ whose (undirected) distances in the underlying graph G from every vertex of C are at least $\lfloor k/2 \rfloor + 1$. Since there is no edge joining A and B, it follows that the (undirected) distance between a and b is at least $2\lfloor k/2 \rfloor + 2 \geq k + 1$, which contradicts the fact that the diameter of G is k.

Without loss of generality we therefore may assume that each vertex of A has (undirected) distance at most $\lfloor k/2 \rfloor$ from some vertex in C. Thus if we consider the digraph D, we may assume that each vertex of A has (directed) distance at most $\lfloor k/2 \rfloor$ from some vertex in C. It follows that the union of all vertices at (directed) distance at most $\lfloor k/2 \rfloor$ from vertices of C contains the entire set A, and therefore

$$|A| \le |C|(d + d^2 + \ldots + d^{\lfloor k/2 \rfloor}) = |C| \; \frac{d(d^{\lfloor k/2 \rfloor} - 1)}{d-1}.$$

From the fact that $A = V(D) \setminus (B \cup C)$ and $|B| \le \frac{2}{3}n$, $|C| \le 2k+1$ we obtain $|A| \ge n - \frac{2}{3}n - (2k+1)$. Combining the inequalities gives

$$\frac{1}{3}n - (2k+1) \le |A| \le (2k+1) \; \frac{d(d^{\lfloor k/2 \rfloor} - 1)}{d-1},$$

thus

$$n \le (6k+3)\frac{(d^{\lfloor k/2 \rfloor + 1} - 1)}{d-1}.$$

□

Theorem 3. Let D be a planar digraph of maximum out-degree and in-degree d and diameter at most k. Then

$$|V(D)| \ge 2\frac{(d-1)^{\lfloor k/2 \rfloor + 1} - 1}{d-2}.$$

Proof. The lower bound comes from a digraph constructed as a natural generalisation of the largest planar digraph of diameter 2 (see Fig 4). The digraph is shown in Fig. 6 and it can be easily seen that the digraph has maximum out-degree and in-degree d, diameter k, and order

$$\begin{aligned} n &= 2(1 + (d-1) + (d-1)(d-1) + (d-1)(d-1)^2 + \ldots + (d-1)(d-1)^{k-1}) \\ &= 2\frac{(d-1)^{\lfloor k/2 \rfloor + 1} - 1}{d-2}. \end{aligned}$$

□

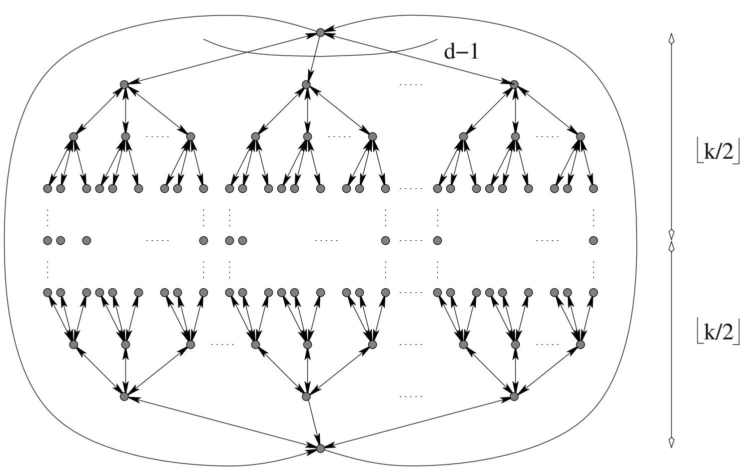


Fig. 6. Large planar digraph of maximum out-degree and in-degree d and diameter k

If we denote by $n^*(S_0)_{d,k}$ the order of largest digraph of given diameter k, maximum out-degree and in-degree d then Theorem 2 and 3 suggest that it is worthwhile to study the limit

$$\lim_{d\to\infty} \frac{n^*(S_0)_{d,k}}{d^{\lfloor k/2 \rfloor}} .$$

The existence of the limit is still not known; if it exists then the value of the limit is between 2 and $6k+3$.

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(a, d)-Edge-Antimagic Total Labelings of Caterpillars

K.A. Sugeng[1], M. Miller[1], Slamin[2], and M. Bača[3]

[1] School of Information Technology and Mathematical Sciences,
University of Ballarat, Victoria 3353, Australia
{k.sugeng, m.miller}@ballarat.edu.au
[2] FKIP, Universitas Jember, Indonesia
slamin@unej.ac.id
[3] Department of Appl. Mathematics,
Technical University, Letná 9, 042 00 Košice, Slovak Republic
Martin.Baca@tuke.sk

Abstract. For a graph $G = (V, E)$, a bijection g from $V(G) \cup E(G)$ into $\{1, 2, ..., |V(G)| + |E(G)|\}$ is called (a, d)-edge-antimagic total labeling of G if the edge-weights $w(xy) = g(x) + g(y) + g(xy)$, $xy \in E(G)$, form an arithmetic progression with initial term a and common difference d. An (a, d)-edge-antimagic total labeling g is called super (a, d)-edge-antimagic total if $g(V(G)) = \{1, 2, ..., |V(G)|\}$.

We study super (a, d)-edge-antimagic total properties of stars S_n and caterpillar $S_{n_1,n_2,...,n_r}$.

1 Introduction

All graphs consider here are finite, simple, and undirected. The graph G has vertex set $V(G)$ and edge set $E(G)$. Let $|V(G)| = v$ and $|E(G)| = e$. For a further general graph theoretic notions, see [12] and [13].

By a *labeling* we mean a one-to-one mapping that carries a set of graph elements into a set of numbers (usually integers), called *labels*.

The *edge-weight* of an edge xy under a labeling is the sum of labels (if present) carried by that edge and the vertices x, y incident with xy.

This paper deals with a notion of (a, d)-edge-antimagic total labeling. For an (a, d)-edge-antimagic total labeling we label all vertices and edges with the numbers from 1 to $v + e$ and we require that the edge-weights form an arithmetic progression with difference d. More formally, we have:

Definition 1. *A bijection $g : V(G) \cup E(G) \to \{1, 2, ..., v + e\}$ is called an (a, d)-edge-antimagic total labeling of G if the set of edge-weights of all edges in G is equal to $\{a, a + d, ..., a + (e - 1)d\}$, for two integers $a > 0$ and $d \geq 0$.*

An (a, d)-edge-antimagic total labeling g is called *super* (a, d)-edge-antimagic total if $g(V(G)) = \{1, 2, ..., v\}$ and $g(E(G)) = \{v + 1, v + 2, ..., v + e\}$.

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These labelings are natural extensions of the notion of edge-magic labeling which was introduced by Kotzig and Rosa [10,11] and the notion of super edge-magic labeling which was defined by Enomoto *et al.* in [6].

By an (a,d)*-edge-antimagic vertex labeling* we mean one-to-one mapping from $V(G)$ into $\{1,2,...,v\}$ such that the set of edge-weights of all edges in G is $\{a, a+d, ..., a+(e-1)d\}$, where $a>0$ and $d \geq 0$ are two fixed integers.

Additionally, in 1989 Hegde [8] introduced the concept of a *strongly* (k,d)*-indexable labeling* which is equivalent to (a,d)-edge-antimagic vertex labeling. Readers are refered to [1] and [9] for more background on strongly (k,d)-indexable labelings.

Many other researchers investigated different forms of antimagic graphs. For example, see Bodendiek and Walther [4,5] and Hartsfield and Ringel [7].

Some relationships between (a,d)-edge-antimagic vertex-labeling, (a,d)-edge-antimagic total labeling and other labelings, namely, edge-magic vertex-labeling and edge-magic total labeling are presented in [2].

Super (a,d)-edge-antimagic total labelings for wheels, fans, complete graphs and complete bipartite graphs can be found in [3].

In this paper we study super (a,d)-edge-antimagic total properties of stars S_n, $n \geq 1$, and caterpillars $S_{n_1,n_2,...,n_r}$, $n_i \geq 1$ for $i = 1,2,...,r$.

2 Lemmas

In this section we present two lemmas which will be useful in the following sections.

Lemma 1. Let $\mathfrak{A}$ be a sequence $\mathfrak{A} = \{c, c+1, c+2, ...c+k\}$, k even. Then there exists a permutation $\Pi(\mathfrak{A})$ of the elements of $\mathfrak{A}$ such that $\mathfrak{A} + \Pi(\mathfrak{A}) = \{2c+\frac{k}{2}, 2c+\frac{k}{2}+1, 2c+\frac{k}{2}+2, ..., 2c+\frac{3k}{2}-1, 2c+\frac{3k}{2}\}$.

Proof. Let $\mathfrak{A}$ be a sequence $\mathfrak{A} = \{a_i | \ a_i = c+(i-1), \ 1 \leq i \leq k+1\}$ and k be even. Define a permutation $\Pi(\mathfrak{A}) = \{b_i | \ 1 \leq i \leq k+1\}$ of the elements of $\mathfrak{A}$ as follows:

$$b_i = \begin{cases} c+\frac{k}{2}+\frac{1-i}{2} & \text{if } i \text{ is odd}, 1 \leq i \leq k+1 \\ c+k+\frac{2-i}{2} & \text{if } i \text{ is even}, 2 \leq i \leq k. \end{cases}$$

By direct computation we obtain that

$\mathfrak{A} + \Pi(\mathfrak{A}) = \{a_i + b_i | \ 1 \leq i \leq k+1\} =$

$\{2c+\frac{k}{2}+\frac{i-1}{2} | \ i \text{ odd}, 1 \leq i \leq k+1\} \cup \{2c+k+\frac{i}{2} | \ i \text{ even}, 2 \leq i \leq k\} =$

$\{2c+\frac{k}{2}, 2c+\frac{k}{2}+1, ..., 2c+\frac{3k}{2}-1, 2c+\frac{3k}{2}\}$

and we arrive at the desired result. □

Lemma 2. Let $\mathfrak{P}$ be a sequence $\mathfrak{P} = \{c, c+1, c+2, ...c+\frac{k-3}{2}, c+\frac{k-1}{2}, c+\frac{k+3}{2}, c+\frac{k+5}{2}, ..., c+k+1\}$, k odd. Then there exists a sequence $\mathfrak{R}$ of the integers $\{1,2,3,...,k+1\}$ such that the sequence $\mathfrak{P}+\mathfrak{R}$ consists of consecutive integers.

Proof. Suppose that k is odd, $k \geq 1$, and consider the sequence $\mathfrak{P} = \{p_i |\ p_i = c-1+i,\ 1 \leq i \leq \frac{k+1}{2}\} \cup \{p_i |\ p_i = c+i,\ \frac{k+3}{2} \leq i \leq k+1\}$. We will distinguish three cases.

Case 1. $k+1 \equiv 2 \pmod 6$.
For $k \geq 1$ we define the sequence $\mathfrak{R} = \{r_i |\ 1 \leq i \leq k+1\}$ as follows.

$$r_i = \begin{cases} k+1-2i \text{ if } i \equiv 1 \pmod 3 & \text{and } 1 \leq i < \frac{k-1}{2} \\ k+1-2i \text{ if } i \equiv 2 \pmod 3 & \text{and } 2 \leq i < \frac{k-1}{2} \\ k+4-2i \text{ if } i \equiv 0 \pmod 3 & \text{and } 3 \leq i \leq \frac{k-1}{2} \end{cases}$$

$$r_i = \begin{cases} k+1 \text{ if } i = \frac{k+1}{2} \\ k \qquad \text{if } i = \frac{k+3}{2} \end{cases}$$

$$r_i = \begin{cases} 2k+1-2i \text{ if } i \equiv 0 \pmod 3 & \text{and } \frac{k+5}{2} \leq i \leq k-1 \\ 2k+1-2i \text{ if } i \equiv 1 \pmod 3 & \text{and } \frac{k+7}{2} \leq i \leq k \\ 2k+4-2i \text{ if } i \equiv 2 \pmod 3 & \text{and } \frac{k+9}{2} \leq i \leq k+1 \end{cases}$$

Case 2. $k+1 \equiv 4 \pmod 6$.
For $k \geq 3$ use the following sequence $\mathfrak{R} = \{r_i |\ 1 \leq i \leq k+1\}$.

$$r_i = \begin{cases} k \qquad \text{if } i = 1 \\ k+1 \text{ if } i = \frac{k+1}{2} \end{cases}$$

$$r_i = \begin{cases} k+4-2i \text{ if } i \equiv 1\ (mod\ 3) \text{ and } 4 \leq i \leq \frac{k-1}{2} \\ k+1-2i \text{ if } i \equiv 2\ (mod\ 3) \text{ and } 2 \leq i < \frac{k-1}{2} \\ k+1-2i \text{ if } i \equiv 0\ (mod\ 3) \text{ and } 3 \leq i < \frac{k-1}{2} \end{cases}$$

$$r_i = \begin{cases} 2k+4-2i \text{ if } i \equiv 1\ (mod\ 3) \text{ and } \frac{k+5}{2} \leq i \leq k+1 \\ 2k+1-2i \text{ if } i \equiv 2\ (mod\ 3) \text{ and } \frac{k+7}{2} \leq i \leq k-1 \\ 2k+1-2i \text{ if } i \equiv 0\ (mod\ 3) \text{ and } \frac{k+3}{2} \leq i \leq k \end{cases}$$

Case 3. $k+1 \equiv 0\ (mod\ 6)$.
For $k \geq 5$ we construct the sequence $\mathfrak{R} = \{r_i |\ 1 \leq i \leq k+1\}$ in the following way.

$$r_i = \begin{cases} k \qquad \text{if } i = 1 \\ k-2 \text{ if } i = 2 \\ k+1 \text{ if } i = \frac{k+1}{2} \\ k-3 \text{ if } i = \frac{k+3}{2} \\ k-1 \text{ if } i = \frac{k+5}{2} \\ k-4 \text{ if } i = \frac{k+7}{2} \end{cases}$$

$$r_i = \begin{cases} k+1-2i \text{ if } i \equiv 1 \pmod 3 & \text{and } 4 \leq i < \frac{k-1}{2} \\ k+4-2i \text{ if } i \equiv 2 \pmod 3 & \text{and } 5 \leq i \leq \frac{k-1}{2} \\ k+1-2i \text{ if } i \equiv 0 \pmod 3 & \text{and } 3 \leq i < \frac{k-1}{2} \end{cases}$$

$$r_i = \begin{cases} 2k+1-2i \text{ if } i \equiv 1 \pmod 3 & \text{and } \frac{k+9}{2} \leq i \leq k-1 \\ 2k+1-2i \text{ if } i \equiv 2 \pmod 3 & \text{and } \frac{k+11}{2} \leq i \leq k \\ 2k+4-2i \text{ if } i \equiv 0 \pmod 3 & \text{and } \frac{k+13}{2} \leq i \leq k+1 \end{cases}$$

It is not difficult to check that in all considered cases the sequence $\mathfrak{P} + \mathfrak{R}$ consists of consecutive integers. □

3 Stars

Let x_o denote the central vertex of star S_n, $n \geq 1$, and x_i, $1 \leq i \leq n$, be its leaves. Our first result provides an upper bound on the parameter d for super (a, d)-edge-antimagicness of star S_n.

Theorem 1. If the star S_n, $n \geq 1$, is super (a,d)-edge-antimagic total then $d \leq 3$.

Proof. Assume that there exists a bijection $g : V(S_n) \cup E(S_n) \to \{1, 2, ..., 2n+1\}$ which is super (a, d)-edge-antimagic total and $W = \{w(xy)|w(xy) = g(x) + g(y) + g(xy), xy \in E(S_n)\} = \{a, a + d, a + 2d, ..., a + (n - 1)d\}$ is the set of edge-weights. It is easy to see that the minimum possible edge-weight in super (a, d)-edge-antimagic total labeling is at least $n + 5$. Consequently, $a \geq n + 5$. On the other hand, the maximum edge-weight is no more than $4n + 2$. Thus

$$a + (n - 1)d \leq 4n + 2$$

and

$$d \leq \frac{4n + 2 - n - 5}{n - 1} = 3\,.$$ □

Theorem 2. Every star S_n, $n \geq 1$, has $(a, 1)$-edge-antimagic vertex-labeling.

Proof. Let f: $V(S_n) \to \{1, 2, ..., n+1\}$ be a vertex-labeling of S_n. A set of edge-weights consists of the consecutive integers $\{a, a + 1, a + 2, ..., a + n - 1\}$ if and only if the values of the vertices x_i, $1 \leq i \leq n$, receive consecutive integers. Then, clearly, the value of the central vertex can be $f(x_0) = 1$ or $f(x_0) = n+1$ and the set of edge-weights is presented as $\{3, 4, ..., n+2\}$ or $\{n+2, n+3, ..., 2n+1\}$. □

In light of the preceding theorem the following result follows immediately.

Theorem 3. The star S_n, $n \geq 1$, has super $(a, 0)$-edge-antimagic total labeling and super $(a, 2)$-edge-antimagic total labeling.

Proof. We consider the $(3, 1)$ or $(n + 2, 1)$-edge-antimagic vertex-labeling from Theorem 2. If we complete the edge-labeling with values in the set $\{n + 2, n + 3, ..., 2n + 1\}$ then we can obtain

(a) super $(2n + 4, 0)$ or $(3n + 3, 0)$-edge-antimagic total labeling, or
(b) super $(n + 5, 2)$ or $(2n + 4, 2)$-edge-antimagic total labeling. □

Theorem 4. The star S_n, $n \geq 1$, has super $(a, 1)$-edge-antimagic total labeling.

Proof. If n is odd we consider the $(3, 1)$ or $(n + 2, 1)$-edge-antimagic vertex-labeling from Theorem 2, where the set of edge-weights gives the sequence $\mathfrak{A} = \{c, c + 1, c + 2, ..., c + k\}$ for $c = 3$ or $c = n + 2$ and $k = n - 1$. From Lemma 1 it follows that there exists a permutation $\Pi(\mathfrak{A})$ of the elements of $\mathfrak{A}$ such that $\mathfrak{A} + [\Pi(\mathfrak{A}) - c + n + 2] = \{c + \frac{3n+1}{2} + 1, c + \frac{3n+1}{2} + 2, ..., c + \frac{5n+1}{2} - 1, c + \frac{5n+1}{2}\}$.

If $[\Pi(\mathfrak{A}) - c + n + 2]$ is the edge-labeling of S_n, then $\mathfrak{A} + [\Pi(\mathfrak{A}) - c + n + 2]$ gives the set of edge-weights of S_n which implies that the total labeling is super $(a, 1)$-edge-antimagic for $a = c + \frac{3n+1}{2} + 1$ and $c = 3$ or $c = n + 2$.

If n is even, define the vertex-labeling $f : V(S_n) \to \{1, 2, ..., n + 1\}$ in the following way.

$$f(x_0) = \frac{n}{2} + 1$$

$$f(x_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} \\ i + 1 & \text{if } \frac{n}{2} + 1 \leq i \leq n. \end{cases}$$

Denote the set of edge-weights of all edges in S_n under vertex-labeling f by $\mathfrak{P} = \{c, c + 1, c + 2, ..., c + \frac{k-3}{2}, c + \frac{k-1}{2}, c + \frac{k+3}{2}, c + \frac{k+5}{2}, ..., c + k + 1\}$ where $c = \frac{n}{2} + 2$, $k = n - 1$.

From Lemma 2 it follows that there exists a sequence of the consecutive integers $\mathfrak{R} = \{1, 2, 3, ..., k + 1\}$ such that the sequence $\mathfrak{P} + [\mathfrak{R} + n + 1]$ consists of consecutive integers. If $[\mathfrak{R} + n + 1]$ is the edge-labeling of S_n then $\mathfrak{P} + [\mathfrak{R} + n + 1]$ describes the set of edge-weights of S_n and we can see that the total labeling of S_n is super $(a, 1)$-edge-antimagic. □

To completely characterize the super (a, d)-edge-antimagic total labeling of S_n, it only remains to consider the case $d = 3$.

Theorem 5. For the star S_n, $n \geq 3$, there is no super $(a, 3)$-edge-antimagic total labeling.

Proof. Assume that S_n, $n \geq 1$, is super $(a, 3)$-edge-antimagic total with a total labeling $g : V(S_n) \cup E(S_n) \to \{1, 2, ..., 2n + 1\}$. In the computation of the edge-weights of S_n, the label of the central vertex x_0 is used n-times, the labels of all the other vertices and the labels of edges are used each once. Thus we have

$$\sum_{i=1}^{n+1} i + (n - 1)g(x_0) + \sum_{j=1}^{n} (n + 1 + j) = \sum_{j=1}^{n} (a + 3(j - 1)) \tag{1}$$

which is obviously equivalent to the equation

$$n^2 + 9n + 2 + 2(n - 1)g(x_0) = 2na. \tag{2}$$

The minimum possible edge-weight is $a = n + 3 + g(x_0)$ if $g(x_0) > 1$ and $a = n + 5$ if $g(x_0) = 1$. From equation (2) it follows that

$$n^2 + 9n + 2 + 2(n - 1)g(x_0) \geq 2n(g(x_0) + n + 3) \tag{3}$$

and then

$$0 \geq n^2 - 3n - 2 + 2g(x_0). \tag{4}$$

We can see that (4) has real solutions if $g(x_0) \leq \frac{17}{8}$. If $g(x_0) = 1$ then from (2) it follows that $n \leq 1$. If $g(x_0) = 2$ then from (3) we obtain that $1 \leq n \leq 2$.

Trivialy, S_1 has super $(6,3)$-edge-antimagic total labeling g_1 with $g_1(x_0) = 1$, $g_1(x_1) = 2$ and $g_1(x_0x_1) = 3$.

In the case $n = 2$, label $g_2(x_0) = 2$, $g_2(x_1) = 1$, $g_2(x_2) = 3$, $g_2(x_0x_1) = 4$, $g_2(x_0x_2) = 5$. The labeling g_2 is super $(7,3)$-edge-antimagic total. □

From the previous theorems it follows that

Theorem 6. The star S_n has super (a,d)-edge-antimagic total labeling if and only if either
(i) $d \in \{0,1,2\}$ and $n \geq 1$, or
(ii) $d = 3$ and $1 \leq n \leq 2$.

4 Caterpillars

A *caterpillar* is a graph derived from a path by hanging any number of leaves from the vertices of the path. The caterpillar can be seen as a sequence of stars $S_1 \cup S_2 \cup \cdots \cup S_r$, where each S_i is a star with central vertex c_i and n_i leaves for $i = 1, 2, \ldots, r$, and the leaves of S_i include c_{i-1} and c_{i+1} for $i = 2, 3, \ldots, r-1$.

We denote the caterpillar as $S_{n_1,n_2,\ldots,n_r}$ where the vertex set is $V(S_{n_1,n_2,\ldots,n_r}) = \{c_i | 1 \leq i \leq r\} \cup \bigcup_{i=2}^{r-1} \{x_i^j | 2 \leq j \leq n_i - 1\} \cup \{x_1^j | 1 \leq j \leq n_1 - 1\} \cup \{x_r^j | 2 \leq j \leq n_r\}$ and the edge set is $E(S_{n_1,n_2,\ldots n_r}) = \{c_i c_{i+1} | 1 \leq i \leq r-1\} \cup \bigcup_{i=2}^{r-1} \{c_i x_i^j | 2 \leq j \leq n_i - 1\} \cup \{c_1 x_1^j | 1 \leq j \leq n_1 - 1\} \cup \{c_r x_r^j | 2 \leq j \leq n_r\}$. $|V(S_{n_1,n_2,\ldots,n_r})| = \sum_{i=1}^{r} n_i - r + 2$ and $|E(S_{n_1,n_2,\ldots,n_r})| = \sum_{i=1}^{r} n_i - r + 1$.

Theorem 7. If a caterpillar, with v vertices and $v-1$ edges, $v \geq 2$, is super (a,d)-edge-antimagic total then $d \leq 3$.

Proof. Assume that there exists a super (a,d)-edge-antimagic total labeling $h : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, 2v-1\}$ and $W = \{w(xy) | w(xy) = h(x) + h(y) + h(xy), xy \in E(G)\} = \{a, a+d, a+2d, \ldots, a+(v-2)d\}$ is the set of edge-weights. The minimum (maximum) possible edge-weight under the labeling h is at least (at most) $v+4$ $(4v-2)$, respectively. Thus we have

$$a + (v-2)d \leq 4v - 2$$

and

$$d \leq 3 \ .$$

□

Theorem 8. All caterpillars are super $(a, 0)$-edge-antimagic total and super $(a, 2)$-edge-antimagic total.

Proof. Considering caterpillar G as a bipartite graph, we can draw G in two rows, each row containing vertices from one partite set. Clearly, it is possible to make the drawing so that there are no edge crossings. Let $a_1, a_2, \ldots, a_t$ be the vertices in the first row ordered from left to right and let $b_1, b_2, \ldots, b_{v-t}$ be the vertices in the second row ordered from left to right.

Define the vertex-labeling $\lambda : V(G) \to \{1, 2, \ldots, v\}$ in the following way.

$\lambda(a_i) = i$ for $1 \le i \le t$,

$\lambda(b_j) = t + j$ for $1 \le j \le v - t$.

It is an easy exercise to check that the set of edge-weights $\bigcup\limits_{xy \in E(G)} \{w(xy)\}$ $= \{t+2, t+3, \ldots, t+v\}$. If the edge-labeling with values $v+1, v+2, \ldots, 2v-1$ is completed to vertex-labeling λ then the resulting labeling can be

1. super $(2v+t+1, 0)$-edge-antimagic total (it was proved by Kotzig and Rosa [10], see also [12]) or
2. super $(v+t+3, 2)$-edge-antimagic total. □

Theorem 9. If v is even then a caterpillar with v vertices has super $(a, 1)$-edge-antimagic total labeling.

Proof. Suppose G is a caterpillar with v vertices and v is even. Consider the vertex-labeling $\lambda : V(G) \to \{1, 2, \ldots, v\}$ from the previous theorem which is $(t+2, 1)$-edge-antimagic, where the set of edge-weights gives the sequence $\mathfrak{A} = \{a, a+1, a+2, \ldots, a+k\}$ for $a = t+2$ and $k = v-2$. According to Lemma 1, we know that there exists a permutation $\Pi(\mathfrak{A})$ of the elements of $\mathfrak{A}$ such that $[\Pi(\mathfrak{A}) - a + v + 1] = \{a + \frac{3v}{2}, a + \frac{3v}{2} + 1, a + \frac{3v}{2} + 2, \cdots, a + \frac{5v}{2} - 2\}$.

If $[\Pi(\mathfrak{A}) - a + v + 1]$ is the edge-labeling of G then $\mathfrak{A} + [\Pi(\mathfrak{A}) - a + v + 1]$ determines the set of edge-weights of G and the resulting total labeling is super $(a + \frac{3v}{2}, 1)$-edge-antimagic. □

Theorem 10. There is a super $(a, 1)$-edge-antimagic total labeling for a caterpillar with odd number of vertices.

Proof. Consider the caterpillar $S_{n_1, n_2, \ldots, n_r}$, where number of vertices v is odd. Order vertices of $S_{n_1, n_2, \ldots, n_r}$ to a sequence $c_1, x_1^1, x_1^2, \ldots, x_1^{n_1-1}, c_2, x_2^2, x_2^3, \ldots, x_2^{n_2-1}, c_3, x_3^2, x_3^3, \ldots, x_3^{n_3-1}, c_4, \ldots, c_r, x_r^2, x_r^3, \ldots, x_r^{n_r}$. Now, we describe a vertex-labeling $\lambda : V(S_{n_1, n_2, \ldots, n_r}) \to \{1, 2, \ldots, v\}$ as follows:

Step 1. Choose a vertex from the sequence in position $\frac{v+1}{2} + 1$ and label it by 1.

Case 1a. If the chosen vertex is a central vertex c_i then we label by consecutive values $2, 3, 4, \ldots$ the leaves of star S_{i+1}, except c_i and including c_{i+2}, and continue by labeling the leaves of star S_{i+3}, except c_{i+2} and including c_{i+4}, and so on until the central vertex c_r or leaves of star S_r are labeled.

Note that the leaves of any star S_i are labeled by ordering in sequence i.e., c_{i-1} receives the smallest label and $x_i^{n_i} = c_{i+1}$ the largest one.

Case 1b. If the chosen vertex is x_i^j then we label by consecutive values $2, 3, 4, \ldots$ the vertices $x_i^{j+1}, x_i^{j+2}, \ldots, x_i^{n_i-1}$ and central vertex c_{i+1} and continue by labeling the leaves of star S_{i+2}, except c_{i+1} and including c_{i+3}, and then label the leaves of star S_{i+4}, except c_{i+3} and including c_{i+5}, so on until the central vertex c_r or leaves of star S_r are labeled.

Step 2. We continue by labeling the vertices at the beginning of the sequence of vertices so that:

Case 2a. If the central vertex c_i was chosen in Step 1 and i is odd (i is even), we start by labeling the leaves of star S_1, including c_2, (the leaves of star S_2, including c_1 and c_3), then by labeling the leaves of star S_t, except c_{t-1} and including c_{t+1}, $t = 3, 5, 7, \ldots$ ($t = 4, 6, 8, \ldots$). After labeling the leaves of star S_r or the central vertex c_r we continue by labeling the leaves of star S_2, including c_1 and c_3, (the leaves of star S_1, including c_2) and then by labeling the leaves of star S_p, except c_{p-1} and including c_{p+1}, $p = 4, 6, 8, \ldots$ ($p = 3, 5, 7, \ldots$) so on until the leaves of star S_{i-1}, except c_i, are labeled.

Case 2b. If the vertex x_i^j was chosen in Step 1 and i is odd (i is even), we start by labeling the leaves of star S_2, including c_1 and c_3, (the leaves of star S_1, including c_2), then by labeling the leaves of star S_t, except c_{t-1} and including c_{t+1}, $t = 4, 6, 8, \ldots$ ($t = 3, 5, 7, \ldots$). After labeling the leaves of star S_r or the central vertex c_r we continue by labeling the leaves of star S_1, including c_2, (the leaves of star S_2, including c_1 and c_3) and then by labeling the leaves of star S_p, except c_{p-1} and including c_{p+1}, $p = 3, 5, 7, \ldots$ ($p = 4, 6, 8, \ldots$) and so on until the last vertex before x_i^j is labeled.

It is easy to see that the set of edge-weights of all edges in $S_{n_1,n_2,\ldots,n_r}$ under vertex labeling λ, gives the sequence $\mathfrak{B} = \{a, a+1, a+2, \ldots, a+\frac{k-3}{2}, a+\frac{k-1}{2}, a+\frac{k+3}{2}, a+\frac{k+5}{2}, \ldots, a+k+1\}$ for $k = v-2$. From Lemma 2, it follows that there exists a sequence of the consecutive integers $\mathfrak{R} = \{1, 2, 3, \ldots, k+1\}$, such that the sequence $\mathfrak{B} + [\mathfrak{R} + v]$ consists of consecutive integers.

If $[\mathfrak{R} + v]$ is the edge-labeling of $S_{n_1,n_2,\ldots,n_r}$ then $\mathfrak{B} + [\mathfrak{R} + v]$ describes the set of edge-weights of caterpillar $S_{n_1,n_2,\ldots,n_r}$. It implies that the caterpillar has super $(a, 1)$-edge-antimagic total labeling. □

Let $S_{n_1,n_2,\ldots,n_r}$ be a caterpillar and $N_1 = \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor + 1} n_{2i-1}$ and $N_2 = \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} n_{2i}$, where $\lfloor \frac{r}{2} \rfloor$ denote the greatest integer smaller than or equal to $\frac{r}{2}$. The next theorems give results for super $(a, 3)$-edge-antimagicness of caterpillar $S_{n_1,n_2,\ldots,n_r}$.

Theorem 11. If r is even and $N_1 = N_2$ or $|N_1 - N_2| = 1$ then the caterpillar has super $(a, 3)$-edge-antimagic total labeling.

Proof.
Case 1. $N_1 = N_2$.
We construct a vertex-labeling $\lambda : V(S_{n_1,n_2,\ldots,n_r}) \to \{1, 2, \ldots, v\}$ in the following way. The leaves of S_2 receive odd integers $1, 3, 5, \ldots, 2n_2 - 1$, where $\lambda(c_1) = 1$ and $\lambda(c_3) = 2n_2 - 1$. The leaves of S_4 (except c_3) receive odd integers $2n_2 + 1, 2n_2 + 3, \ldots, 2n_2 + 2n_4 - 3$ where $\lambda(c_5) = 2n_2 + 2n_4 - 3$. This algorithm is applied for labeling the leaves of $S_6, S_8, \ldots, S_r$.

Then the leaves of S_1 are labeled by even integers $2, 4, 6, \ldots, 2n_1$, where $\lambda(c_2) = 2n_1$ and the leaves of S_3 (except c_2) are labeled by even integers $2n_1 + 2, 2n_1 + 4, \ldots, 2n_1 + 2n_3 - 2$ where $\lambda(c_4) = 2n_1 + 2n_3 - 2$. We continue by labeling the leaves of $S_5, S_7, \ldots, S_{r-1}$, where $\lambda(c_r)$ will be the largest even value i.e., v.

It is easy to verify that the labeling λ uses each integer from the set $\{1, 2, 3, \ldots, v\}$ exactly once. By direct computation we obtain that the set of edge-weights consists of an arithmetic sequence $W = \{w(xy) | w(xy) = \lambda(x) + \lambda(y),\ xy \in E(S_{n_1,n_2,\ldots n_r})\} = \{a, a+d, a+2d, \ldots, a+(v-2)d\}$ for $a = 3$ and $d = 2$. If we complete the edge-labeling of $S_{n_1,n_2,\ldots n_r}$ with values in the set $\{v+1, v+2, v+3, \ldots, 2v-1\}$ then we obtain a super $(2v+2, 1)$-edge-antimagic total labeling or a super $(v+4, 3)$-edge-antimagic total labeling.

Case 2. $N_2 = N_1 + 1$.
Consider a super $(v + 4, 3)$-edge-antimagic total labeling f of a caterpillar $S_{n_1,n_2,\ldots n_r}$ from *Case 1* and add one leaf z to c_r. Define a new labeling f_1 as follows:

$$f_1(x) = f(x) \text{ for all vertices } x \in V(S_{n_1,n_2,\ldots n_r}),$$
$$f_1(z) = f(c_r) + 1 = v + 1,$$
$$f_1(xy) = f(xy) + 1 \text{ for all edges } xy \in E(S_{n_1,n_2,\ldots n_r}),$$
$$f_1(c_r z) = 2v + 1.$$

We can see that f_1 is a super $(v + 5, 3)$-edge-antimagic total labeling of $S_{n_1,n_2,\ldots n_r} + \{z\}$.

Case 3. $N_1 = N_2 + 1$.
Now, we consider the vertex labeling $\lambda : V(S_{n_1,n_2,\ldots,n_r}) \to \{1, 2, \ldots, v\}$ described in *Case 1*. Define a new vertex-labeling g such that for any vertex $x \in V(S_{n_1,n_2,\ldots n_r})$

$$g(x) = \lambda(x) + 1 \text{ if } \lambda(x) \text{ is an odd value},$$
$$g(x) = \lambda(x) - 1 \text{ if } \lambda(x) \text{ is an even value},$$

where the largest even label v is received by a vertex x_r^j and $g(c_r) = v - 1$.

Clearly, the values of g are $1, 2, \ldots, v$ and it is not difficult to check that the edge-weights constitute an arithmetic progression $3, 5, 7, \ldots, 3 + 2(v - 2)$.

Connect a leaf z to c_{r-1} and define a vertex-labeling g_1 in the following way:

$$g_1(x) = g(x) \text{ for all vertices } x \in V(S_{n_1,n_2,\ldots n_r}) \text{ except } c_r,$$
$$g_1(z) = v - 1,$$
$$g_1(c_r) = v + 1.$$

Again we can see that the vertex-labeling g_1 uses each integer from the set $\{1, 2, 3, \ldots, v+1\}$ exactly once and the edge-weights constitute an arithmetic progression $a, a+d, a+2d, \ldots, a+(v-1)d$ for $a = 3$ and $d = 2$. If we combine the vertex-labeling g_1 and the edge-labeling $g_1^* : E(S_{n_1,n_2,\ldots,n_r}) \cup \{c_{r-1}z\} \rightarrow \{v+2, v+3, \ldots, 2v+1\}$ then we are able to obtain a super $(v+5, 3)$-edge-antimagic total labeling of $S_{n_1,n_2,\ldots,n_r} + \{z\}$. □

Theorem 12. If r is odd and $N_1 = N_2$ or $N_1 = N_2 + 1$, then the caterpillar has super $(a, 3)$-edge-antimagic total labeling.

Proof. Let $S_{n_1,n_2,\ldots,n_r}$ be a caterpillar and r be odd.
Case 1. $N_1 = N_2$.
Construct a vertex-labeling $\lambda : V(S_{n_1,n_2,\ldots,n_r}) \rightarrow \{1, 2, \ldots, v\}$ by the algorithm from *Case 1* of the previous theorem such that the leaves of stars $S_2, S_4, \ldots, S_{r-1}$ receive odd integers $1, 3, 5, \ldots, 2n_2 - 1, \ldots$ and the leaves of stars $S_1, S_3, S_5, \ldots, S_r$ receive even integers $2, 4, 6, \ldots, 2n_1, \ldots,$ where the central vertex c_r receives the largest odd value.

The edge-weights, under vertex-labeling λ, form an arithmetic progression with initial term 3 and common difference $d = 2$. If we combine the vertex-labeling λ and the edge-labeling $\lambda^* : E(S_{n_1,n_2,\ldots,n_r}) \rightarrow \{v+1, v+2, \ldots, 2v-1\}$ then the resulting labeling is super $(2v+2, 1)$-edge-antimagic total or super $(v+4, 3)$-edge-antimagic total.

Case 2. $N_1 = N_2 + 1$.
Consider the super $(v+4, 3)$-edge-antimagic total labeling f of a caterpillar $S_{n_1,n_2,\ldots,n_r}$ from the previous case. Add one leaf z to c_r and define a new labeling f_1 as in the *Case 2* of previous theorem. It can be seen that f_1 is super $(v+5, 3)$-edge-antimagic total labeling of $S_{n_1,n_2,\ldots,n_r} + \{z\}$. □

Regarding the previous theorems, a double star $S_{m,n}$, $m, n \geq 2$, can have $(a, 3)$-edge-antimagic total labeling if $m = n$ or $|m - n| = 1$. For other cases the following theorem proves that a double star cannot have $(a, 3)$-edge-antimagic total labeling.

Theorem 13. For the double star $S_{m,n}, m \neq n$ and $|m - n| \neq 1$, there is no super $(a, 3)$-edge-antimagic total labeling.

Proof. Assume that $S_{m,n}$ has a super $(a, 3)$-edge-antimagic total labeling $g : V(S_{m,n}) \cup E(S_{m,n}) \rightarrow \{1, 2, \ldots, 2m+2n-1\}$ and $W = \{w(xy) | xy \in E(S_{m,n})\}$ $= \{a, a+3, a+6, \ldots, a+(m+n-2)3\}$ is the set of the edge-weights. The sum of the edge weights in the set W is

$$\sum_{xy \in E(S_{m,n})} w(xy) = (m+n-1)a + \frac{3}{2}(m+n-1)(m+n-2). \qquad (5)$$

The minimum possible edge-weight under labeling g is at least $m+n+4$. Consequently $a \geq m+n+4$. On the other hand, the maximum possible edge-weight is at most $4m+4n-2$ and

$$a + (m+n-2)3 \leq 4m + 4n - 2$$

and

$$a \leq m + n + 4.$$

Thus $a = m + n + 4$. In the computation of the edge-weights of $S_{m,n}$ the label of the central vertex c_1 (c_2) is used m times (n times) and the labels of the leaves and values of the edges are used once. The sum of all vertex labels and edge labels used to calculate the edge-weights is thus equal to

$$\sum_{i=1}^{2m+2n-1} i + (m-1)g(c_1) + (n-1)g(c_2). \tag{6}$$

From (5) and (6) we have

$$m^2 + n^2 + 2mn - m - n - 2 = 2(m-1)g(c_1) + 2(n-1)g(c_2). \tag{7}$$

Since $a = m + n + 4$ and $a + (m + n - 2)3 = 4m + 4n - 2$, then the vertices labeled by 1 and 2 have to be adjacent, and so do the vertices labeled by $m + n$ and $m + n - 1$. This leads to the following cases:

Case 1. If $g(c_1) = 1$ and $g(c_2) = m + n$ then from (7) it follows that $m^2 - m = n^2 - n$ but this is impossible since $m \neq n$.

Case 2. If $g(c_1) = 1$ and $g(c_2) = m + n - 1$ then from (7) it follows that $m^2 - m - 2 = n^2 - 3n$. It is true if $n = m + 1$ but this contradicts $|m - n| \neq 1$.

Case 3. Let $g(c_1) = 2$ and $g(c_2) = m + n$. From (7) we have the equation $m^2 - 3m = n^2 - n - 2$ which is correct for $m = n + 1$ but this contradicts $|m - n| \neq 1$.

Case 4. Assume that $g(c_1) = 2$ and $g(c_2) = m + n - 1$. From (7) follows that $m^2 - 3m = n^2 - 3n$ which is a contradiction to $m \neq n$. □

Concluding this paper, let us pose the following.

Conjecture 1. For the caterpillar $S_{n_1,n_2,\ldots,n_r}$, $N_1 \neq N_2$ and $|N_1 - N_2| \neq 1$, there is no super $(a, 3)$-edge-antimagic total labeling.

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An Upper Bound for the Ramsey Number of a Cycle of Length Four Versus Wheels

Surahmat[1,2,⋆], Edy Tri Baskoro[1], Saladin Uttunggadewa[1], and Hajo Broersma[2]

[1] Department of Mathematics, Institut Teknologi Bandung (ITB), Jalan Ganesa 10 Bandung 40132, Indonesia
{kana_s, ebaskoro, s_uttunggadewa}@dns.math.itb.ac.id

[2] Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands
broersma@math.utwente.nl

Abstract. For given graphs G and H, the *Ramsey number* $R(G, H)$ is the smallest positive integer n such that every graph F of n vertices satisfies the following property: either F contains G or the complement of F contains H. In this paper, we show that the Ramsey number $R(C_4, W_m) \le m + \lceil \frac{m}{3} \rceil + 1$ for $m \ge 6$.

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1 Introduction

Throughout the paper, all graphs are finite and simple. For given graphs G and H, the *Ramsey number* $R(G, H)$ is the smallest integer n such that any graph F of order n satisfies the following property: either F contains G or $\overline{F}$ contains H, where $\overline{F}$ is the complement of F. For each $x \in V(G)$ and $S \subseteq V(G)$, define $N_S(x) = \{y \in S : xy \in E(G)\}$ and $G[S]$ denotes the subgraph induced by S in G. A *cycle* of n vertices is denoted by C_n. A *wheel* $W_n = [x, C_n]$ with $(n+1)$ vertices is a graph by a vertex x, called a *hub* of the wheel, adjacent to all vertices of C_n, called a *rim*. A *cocktail-party graph* H_n is a graph obtained from a complete graph of $2n$ vertices by removing n independent edges.

A graph F will be called a $(G, H)-$*good graph* if F contains no G as subgraph and $\overline{F}$ contains no H as subgraph. Any $(G, H)-$good graph on n vertices will be called a $(G, H, n)-$*good graph.* Thus, the Ramsey number $R(G, H) = n$ if there is no (G, H, n)-digraph.

Graph Ramsey theory has grown enormously in the last three decades to become presently one of the most active areas in Ramsey theory. One of the most general results in graph Ramsey theory is the following. For a graph G

⋆ Permanent address: Department of Mathematics Education UNISMA, Jalan MT Haryono 193 Malang 65144, Indonesia.

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(with no isolated vertices), let $\chi(G)$ denote the chromatic number of G and let $c(G)$ denote the cardinality of the largest connected component of G. Then, Chvátal and Harary [3] showed: $R(G,H) \geq (\chi(G)-1)(c(H)-1)+1$. In particular, Chvátal [2] obtained $R(K_m,T_n) = (m-1)(n-1)+1$, where T_n is an arbitrary tree on n vertices and K_m is a complete graph on m vertices.

Many other interesting results on graph Ramsey numbers have been obtained, see [4] for a nice survey. In particular, for a combination of cycles and wheels, Burr and Erdös [1] showed that $R(C_3,W_m) = 2m+1$ for any $m \geq 5$. Radziszowski and Xia [5] gave a simple and unified method to prove the Ramsey number $R(C_3,G)$ where G is either a path, a cycle or a wheel. For more general result, Zhou [8] obtained $R(C_n,W_m) = 2m+1$ for odd n and $m \geq 5n-7$. Recently, two papers of [6] and [7] showed independently that

Theorem 1. [6] *$R(C_4,W_4) = 9$, $R(C_4,W_5) = 10$, and $R(C_4,W_6) = 9$.*

Theorem 2. [7] *$R(C_4,W_m) = 11, 12, 13, 14, 16$ and 17, for $7 \leq m \leq 12$, respectively.*

However, the finding of the Ramsey number $R(C_4,W_m)$, $m \geq 13$, is still an open problem.

In this paper, we present an upper bound for the Ramsey numbers of cycle C_4 versus wheels in a more general situation. The main result of this paper is the following.

Theorem 3. *$R(C_4,W_m) \leq m + \lceil \frac{m}{3} \rceil + 1$ for $m \geq 6$.*

2 Proof of the Main Result

In order to prove Theorem 3, we need the following three lemmas.

Lemma 1. *Let G be a graph with n vertices such that $\overline{G}$ contains a wheel $W_{m-1} = [a_0, A]$ where a_0 is a hub and $A = \{a_1, a_2, \ldots, a_{m-1}\}$ is a rim and G contains no C_4. If $\overline{G}$ contains no W_m then $|N_A(x)| \geq m-2$ for any $x \in V(G) \setminus V(W_{m-1})$ with $xa_0 \notin E(G)$.*

Proof. Suppose $|N_A(x)| < m-2$ for some $x \in V(G)\backslash A$ with $xa_0 \notin E(G)$. Let $xa_i, xa_j \notin E(G)$ for $1 \leq i < j \leq m-1$. Then, x must be adjacent to all vertices in $\{a_{i-1}, a_{i+1}, a_{j-1}, a_{j+1}\}$ since $\overline{G} \not\supseteq W_m$. This implies $a_{i-1}a_{j-1}$, $a_{i+1}a_{j+1} \in E(G)$. Since $G \not\supseteq C_4$, we have $a_{i-1}a_{i+1}$, $a_{i+1}a_{j-1}$, $a_{j-1}a_{j+1}, a_{j+1}a_{i-1} \notin E(G)$. This implies that a_i is adjacent to all vertices in $\{a_{j-1}, a_{j+1}\}$. Thus, we have $G \supseteq C_4$ with $V(C_4) = \{x, a_{j-1}, a_i, a_{j+1}\}$, a contradiction. □

Lemma 2. *Let G be a graph with n vertices such that $\overline{G}$ contains a wheel $W_{m-1} = [a_0, A]$ where a_0 is a hub and $A = \{a_1, a_2, \ldots, a_{m-1}\}$ is a rim and G contains no C_4. Let $B = \{x \in V(G) \setminus V(W_{m-1}) : xa_0 \notin E(G)\}$. If $\overline{G}$ contains no W_m then $|B| \leq 1$.*

Proof. By Lemma 1 we have $|N_A(x)| \geq m-2$ for each $x \in B$. Since $G \not\supseteq C_4$ we have $|B| \leq 1$. $\square$

Lemma 3. *Let G be a graph with n vertices such that $\overline{G}$ contains a wheel W_{m-1} where a_0 is a hub and $A = \{a_1, a_2, \ldots, a_{m-1}\}$ and G contains no C_4. Let $B = \{x \in V(G) \setminus V(W_{m-1}) : xa_0 \notin E(G)\}$ and $D = \{x \in V(G) \setminus V(W_{m-1}) : xa_0 \in E(G)\}$. If $\overline{G}$ contains no W_m and $|B| = 1$ then $|D| \leq 2$.*

Proof. Suppose to the contrary that $\{x_1, x_2, x_3\} \subseteq D$. Let $y_1 \in B$. By Lemma 1 we have $|N_A(y_1)| \geq m-2$. Choose $a_1 \in A$ so that $A_1 = A \setminus \{a_1\}$ consists of all vertices adjacent to y_1. Since $G \not\supseteq C_4$ we have $\overline{G}[A] \supseteq F$ where $F = H_{\frac{m-2}{2}}$ for m even or $F = H_{\frac{m-3}{2}} + K_1$ for m odd. Since G contains no C_4 then a_1 is adjacent to at most one vertex in $\{x_1, x_2, x_3\}$. Let a_1 be not adjacent to x_1 and x_2. Since a_1 is adjacent to at most one vertex in A_1 and it is easy to get W_m in $\overline{G}[A_1 \cup \{a_0, x_1, x_2\}]$ with a_1 as a hub, a contradiction. $\square$

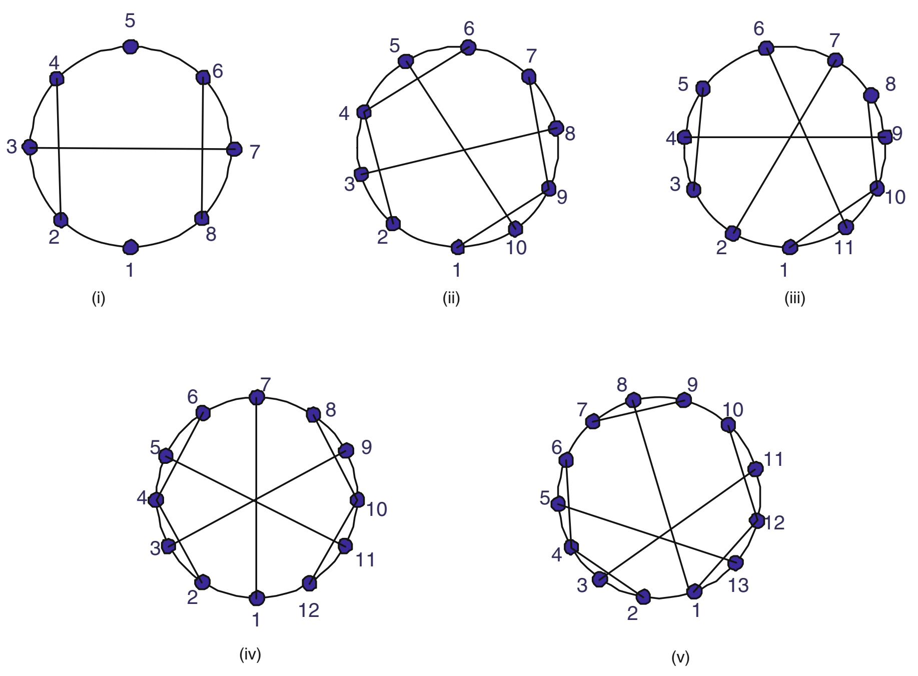

Fig. 1. (i) $(C_4, W_6, 8)-$good graph (ii) $(C_4, W_7, 10)-$good graph (iii) $(C_4, W_8, 11)-$good graph (iv) $(C_4, W_9, 12)-$good graph (v) $(C_4, W_{10}, 13)-$good graph

Proof of Theorem 3. We shall use induction on m. For $6 \leq k \leq 12$, by Theorem 1 and Theorem 2 we have $R(C_4, W_k) = k + \lceil \frac{k}{3} \rceil + 1$. Futhermore, we also obtain the $(C_4, W_k)-$good graphs of for $k = 6, 7, 8, 9, 10$ in the following figures. Assume that the theorem holds for $m = k-1 \geq 12$, namely $R(C_4, W_m) \leq m + \lceil \frac{m}{3} \rceil + 1$.

Let G be a graph of order $k + \lceil \frac{k}{3} \rceil + 1$ containing no a C_4. We shall show that $\overline{G}$ contains W_k. By the assumption of induction, $\overline{G}$ must contain a wheel W_{k-1}. Let a_0 be the hub of W_{k-1} and $A = \{a_1, a_2, \ldots, a_{k-1}\}$ be the rim of W_{k-1}. For a contradiction, suppose $\overline{G}$ contains no W_k. By Lemma 2, we have $|B| \leq 1$ with $B = \{x \in V(G) \setminus V(W_{m-1}) : xa_0 \notin E(G)\}$. Let $D = \{x \in V(G) \setminus V(W_{m-1}) : xa_0 \in E(G)\}$. Now, we distinguish two cases in the following.

Case 1. $|B| = 1$.
We have $|D| = k + \lceil \frac{k}{3} \rceil + 1 - |V(W_{k-1})| - |B| = k + \lceil \frac{k}{3} \rceil + 1 - k - 1 = \lceil \frac{k}{3} \rceil \geq 3$, but by Lemma 3 we get $|D| \leq 2$, a contradiction.

Case 2. $|B| = 0$.
We have $|D| = k + \lceil \frac{k}{3} \rceil + 1 - |V(W_{k-1})| - |B| = k + \lceil \frac{k}{3} \rceil + 1 - k - 0 = \lceil \frac{k}{3} \rceil + 1$. Since $|A| = k - 1$ and $k \geq 13$, by the Pigeon Hole principle we have $|N_A(x)| \leq 2$ for some $x \in D$. It is easy to get W_k in $\overline{G}$ with x as a hub since $k \geq 13$ and $\lceil \frac{k}{3} \rceil + 1 \geq 6$, a contradiction.

Therefore, in any case, the theorem holds. □

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Constructions for Nonhamiltonian Burkard-Hammer Graphs

Ngo Dac Tan[1,2] and Chawalit Iamjaroen[2]

[1] Institute of Mathematics,
18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam
ndtan@math.ac.vn

[2] Department of Mathematics, Mahasarakham University,
Kamrieng, Kantarawichai, Mahasarakham 44150, Thailand
chawalit.i@msu.ac.th

Abstract. A graph $G = (V, E)$ is called a split graph if there exists a partition $V = I \cup K$ such that the subgraphs $G[I]$ and $G[K]$ of G induced by I and K are empty and complete graphs, respectively. In 1980, Burkard and Hammer gave a necessary condition for split graphs with $|I| < |K|$ to be hamiltonian. This condition is not sufficient. In this paper, we give two constructions for producing infinite families of split graphs with $|I| < |K|$, which satisfy the Burkard-Hammer condition but have no Hamilton cycles.

1 Introduction

All graphs considered in this paper are finite undirected graphs without loops and multiple edges. If G is a graph, then $V(G)$ and $E(G)$ (or V and E in short) will denote its vertex-set and its edge-set, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ or shortly by $N(S)$ if G is clear from the context. The subgraph of G induced by $W \subseteq V(G)$ is denoted by $G[W]$. If A is a set and $x \in A$, then for convenience we will write $A - x$ instead of $A \setminus \{x\}$. Unless otherwise indicated, our graph-theoretic terminology will follow [1].

A graph $G = (V, E)$ is called a *split graph* if there exists a partition $V = I \cup K$ such that the subgraphs $G[I]$ and $G[K]$ of G induced by I and K are empty and complete graphs, respectively. We will denote such a graph by $S(I \cup K, E)$. Further, a split graph $G = S(I \cup K, E)$ is called a *complete split graph* if any $u \in I$ is adjacent in G to any $v \in K$. We consider the trivial graph consisting of only one vertex v to be the complete split graph $S(I \cup K, E)$ with $I = \emptyset$, $K = \{v\}$ and $E = \emptyset$. The notion of split graphs was introduced in 1977 by Foldes and Hammer [5]. These graphs have been paid attention because they have connection with many combinatorial problems (see [4], [6], [7]). Many generalizations of split graphs have been made. The newest one is the notion of bisplit graphs introduced by A. Brandstädt, P.L. Hammer, V.B. Le and V.V. Lozin [2].

A split graph $G = S(I \cup K, E)$ with $|I| = |K|$ has a Hamilton cycle if and only if the bipartite graph $G' = G - E(G[K])$ has a Hamilton cycle. Further, since no

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Hamilton cycle exists in G if $|I| > |K|$, we can concentrate here on the remaining case $|I| < |K|$. It is not difficult to show that a split graph $G = S(I \cup K, E)$ with $|I| < |K|$ has a Hamilton cycle if and only if the graph $G[I \cup N(I)]$ has a Hamilton cycle. So without loss of generality, we may assume in this paper that any considered split graph $S(I \cup K, E)$ satisfies $|I| < |K|$ and $K = N(I)$.

In 1980, Burkard and Hammer gave a necessary condition for a split graph $G = S(I \cup K, E)$ with $|I| < |K|$ to be hamiltonian [3] (see Section 2 for more details). We will call this condition the *Burkard-Hammer condition*. Also, we will call a split graph $G = S(I \cup K, E)$ with $|I| < |K|$, which satisfies the Burkard-Hammer condition, a *Burkard-Hammer graph*.

Thus, by [3] any hamiltonian split graph $G = S(I \cup K, E)$ with $|I| < |K|$ is a Burkard-Hammer graph. From the results obtained in [8], it is not difficult to show that a Burkard-Hammer graph $G = S(I \cup K, E)$ with the minimum degree $\delta(G) \geq |I| - 2$ has a Hamilton cycle. Thus, the converse is true in this particular case. But in general, it is not true. The first nonhamiltonian Burkard-Hammer graph denoted in this paper by H^1 was indicated in [3]. Recently, further three nonhamiltonian Burkard-Hammer graphs denoted here by H^2, H^3 and H^4 have been discovered in [9]. It also has been shown in [9] that these graphs H^1, H^2, H^3 and H^4 are the only nonhamiltonian graphs among Burkard-Hammer graphs $G = S(I \cup K, E)$ with the minimum degree $\delta(G) \geq |I| - 3$. A natural question raised from this result is whether the number of nonhamiltonian Burkard-Hammer graphs is finite. If the answer to this question is positive, then by finding all of them we can say that the Burkard-Hammer condition is a necessary and sufficient condition for the remaining split graphs $G = S(I \cup K, E)$ with $|I| < |K|$ to be hamiltonian. Unfortunately, this is not the case. In this paper we give two constructions that produce infinite families of nonhamiltonian Burkard-Hammer graphs. We believe that the obtained families will be useful for considering the question if the Burkard-Hammer condition can be sharpened to a necessary and sufficient one, which was posed in [3] by Burkard and Hammer.

2 Burkard-Hammer Graphs

Let $G = S(I \cup K, E)$ be a split graph and $I' \subseteq I, K' \subseteq K$. Denote by $B_G(I', K')$ the graph $G[I' \cup K'] - E(G[K'])$. It is clear that $G' = B_G(I', K')$ is a bipartite graph with the bipartition subsets I' and K'. So we will call $G' = B_G(I', K')$ the bipartite subgraph of G induced by I' and K'. For a component $G'_j = B_G(I'_j, K'_j)$ of $G' = B_G(I', K')$ we define

$I(G'_j) = I'_j = I \cap V(G'_j),\ K(G'_j) = K'_j = K \cap V(G'_j),$

$k_G(G'_j) = k_G(I'_j, K'_j) = \begin{cases} |I'_j| - |K'_j| & \text{if } |I'_j| > |K'_j|, \\ 0 & \text{otherwise.} \end{cases}$

If $G' = B_G(I', K')$ has r components $G'_1 = B_G(I'_1, K'_1), \ldots, G'_r = B_G(I'_r, K'_r)$, then we define

$$k_G(G') = k_G(I', K') = \sum_{j=1}^{r} k_G(G'_j).$$

A component $G'_j = B_G(I'_j, K'_j)$ of $G' = B_G(I', K')$ is called a *T-component* (resp., *H-component, L-component*) if $|I'_j| > |K'_j|$ (resp., $|I'_j| = |K'_j|$, $|I'_j| < |K'_j|$). Let $h_G(G') = h_G(I', K')$ denote the number of H-components of G'.

In 1980, Burkard and Hammer proved the following necessary condition for a split graph $G = S(I \cup K, E)$ to be hamiltonian [3].

Theorem 1 ([3]). Let $G = S(I \cup K, E)$ be a split graph with $|I| < |K|$. If G is hamiltonian, then

$$k_G(I', K') + \max\{1, h_G(I', K')/2\} \leq |N_G(I')| - |K'|$$

holds for any $\emptyset \neq I' \subseteq I, K' \subseteq N_G(I')$ such that $(k_G(I', K'), h_G(I', K')) \neq (0, 0)$.

We will shortly call the condition in Theorem 1 the *Burkard-Hammer condition.* We also call a split graph $G = S(I \cup K, E)$ with $|I| < |K|$, which satisfies the Burkard-Hammer condition, a *Burkard-Hammer graph.* Thus, by Theorem 1, all hamiltonian split graphs $G = S(I \cup K, E)$ with $|I| < |K|$ are Burkard-Hammer graphs. In particular, since a complete split graph $G = S(I \cup K, E)$ with $|I| < |K|$ is hamiltonian, it is a Burkard-Hammer graph, too. But not all Burkard-Hammer graphs are hamiltonian. The following graph, denoted by $H^1 = S(I(H^1) \cup K(H^1),\ E(H^1))$, has been given in [3] (see Figure 1):

$$\begin{aligned}
I(H^1) &= \{u_1, u_2, u_3, u_4, u_5\},\\
K(H^1) &= \{v_1, v_2, v_3, v_4, v_5, v_6\},\\
E(H^1) &= E'(H^1) \cup E''(H^1), \text{ where}\\
E'(H^1) &= \{u_1v_1, u_1v_2, u_2v_2, u_2v_4, u_3v_2, u_3v_3,\\
&\qquad u_3v_6, u_4v_1, u_4v_4, u_4v_6, u_5v_5, u_5v_6\},\\
E''(H^1) &= \{v_iv_j \mid i \neq j \text{ and } i, j \in \{1, 2, 3, 4, 5, 6\}\}.
\end{aligned}$$

This graph can be shown to be a Burkard-Hammer graph, but it has no Hamilton cycles. Recently in [9], Ngo Dac Tan and Le Xuan Hung have discovered further three nonhamiltonian Burkard-Hammer graphs H^2, H^3 and H^4, where

$H^2 = H^1 + u_4v_2$,
$H^3 = H^1 + u_5v_2$ and
$H^4 = H^1 + \{u_4v_2, u_5v_2\}$.

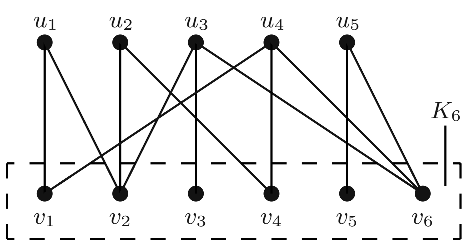

Fig. 1. The graph H^1

In the next two sections we will give two constructions that produce several infinite families of nonhamiltonian Burkard-Hammer graphs. This shows that to obtain a necessary and sufficient condition for split graphs to be hamiltonian we should sharpen properly the Burkard-Hammer condition.

3 Construction 1

Let $G_1 = S(I_1 \cup K_1, E_1)$ and $G_2 = S(I_2 \cup K_2, E_2)$ be split graphs with

$$V(G_1) \cap V(G_2) = \emptyset,$$

and v be any vertex of K_1. We say that a graph G is an *expansion of* G_1 *by* G_2 *at* v if G is the graph obtained from $(G_1 - v) \cup G_2$ by adding the set of edges

$$E_0 = \{x_i v_j \mid x_i \in V(G_1), v_j \in K_2 \text{ and } x_i v \in E_1\}.$$

It is clear that such a graph G is a split graph $S(I \cup K, E)$ with $I = I_1 \cup I_2$, $K = (K_1 - v) \cup K_2$ and is uniquely determined by G_1, G_2 and $v \in K_1$. Because of this, we will denote this graph by $G_1[G_2, v]$.

We say that a graph G is an *expansion of* G_1 *by* G_2 if G is an expansion of G_1 by G_2 at some vertex $v \in K_1$.

If H^1 is the graph defined in Section 2 and $G^* = S(I^* \cup K^*, E^*)$ is the complete split graph with $I^* = \{u_1^*, u_2^*\}$ and $K^* = \{v_1^*, v_2^*, v_3^*\}$, then the expansion $H^1[G^*, v_6]$ is the graph drawn in Figure 2.

We prove now the following results.

Theorem 2. Let $G_1 = S(I_1 \cup K_1, E_1)$ be a Burkard-Hammer graph and $G_2 = S(I_2 \cup K_2, E_2)$ be a complete split graph with $|I_2| < |K_2|$. Then any expansion of G_1 by G_2 is a Burkard-Hammer graph.

Proof. Let $G = G_1[G_2, v]$, where $v \in K_1$. We have noted before that the graph G is a split graph $S(I \cup K, E)$ with $I = I_1 \cup I_2$ and $K = (K_1 - v) \cup K_2$. Since $|I_1| < |K_1|$ and $|I_2| < |K_2|$, we get $|I| < |K|$. We show that G satisfies the Burkard-Hammer condition.

Let $\emptyset \neq I' \subseteq I$. We consider separately two cases.

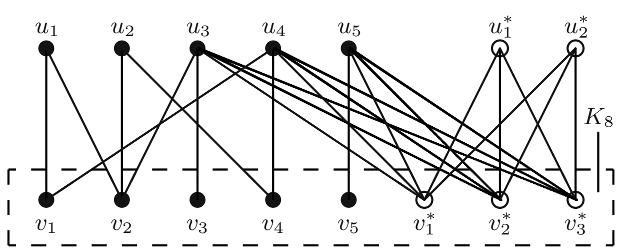

Fig. 2. The expansion $H^1[G^*, v_6]$

Case (i): $I' \subseteq I_2$.
In this case, by the construction of $G = G_1[G_2, v]$, we have $N_G(I') = N_{G_2}(I') \subseteq K_2$. So for any $K' \subseteq N_G(I')$ we have $G' = B_G(I', K') = B_{G_2}(I', K')$. Therefore, $k_G(I', K') = k_{G_2}(I', K')$ and $h_G(I', K') = h_{G_2}(I', K')$. If

$$(k_G(I', K'), h_G(I', K')) = (k_{G_2}(I', K'), h_{G_2}(I', K')) \neq (0, 0),$$

then since G_2 is a Burkard-Hammer graph we have

$$\begin{aligned} k_G(I', K') + \max\{1, h_G(I', K')/2\} &= k_{G_2}(I', K') + \max\{1, h_{G_2}(I', K')/2\} \\ &\leq |N_{G_2}(I')| - |K'| = |N_G(I')| - |K'|. \end{aligned}$$

Thus, the Burkard-Hammer condition holds in Case (i).

Case (ii): $I' \not\subseteq I_2$.
Then $I' = I'_1 \cup I'_2$ with $I'_1 \cap I'_2 = \emptyset$, where $I'_1 = I' \cap I_1 \neq \emptyset$ and $I'_2 = I' \cap I_2$. We again divide this case into two subcases.

Subcase (ii-1): $v \notin N_{G_1}(I'_1)$.
In this subcase, we have $N_G(I') = N_G(I'_1) \cup N_G(I'_2)$ with $N_G(I'_1) \cap N_G(I'_2) = \emptyset$. Moreover, $N_G(I'_1) = N_{G_1}(I'_1)$ and $N_G(I'_2) = N_{G_2}(I'_2)$. Let $K' \subseteq N_G(I')$, $K'_1 = K' \cap (K_1 - v)$ and $K'_2 = K' \cap K_2$. Then $K' = K'_1 \cup K'_2$ with $K'_1 \cap K'_2 = \emptyset$. It is not difficult to see that in this subcase $G' = B_G(I', K')$ is the union of disjoint graphs $B_{G_1}(I'_1, K'_1)$ and $B_{G_2}(I'_2, K'_2)$. So

$$\begin{aligned} k_G(I', K') &= k_{G_1}(I'_1, K'_1) + k_{G_2}(I'_2, K'_2), \\ h_G(I', K') &= h_{G_1}(I'_1, K'_1) + h_{G_2}(I'_2, K'_2). \end{aligned}$$

Therefore, if $(k_G(I', K'), h_G(I', K')) \neq (0, 0)$, then at least one of pairs

$$(k_{G_1}(I'_1, K'_1), h_{G_1}(I'_1, K'_1)) \text{ and } (k_{G_2}(I'_2, K'_2), h_{G_2}(I'_2, K'_2))$$

is not $(0, 0)$.

First assume that both of these pairs are not $(0, 0)$. Since G_1 and G_2 are Burkard-Hammer graphs, we have

$$\begin{aligned} k_{G_1}(I'_1, K'_1) + \max\{1, h_{G_1}(I'_1, K'_1)/2\} &\leq |N_{G_1}(I'_1)| - |K'_1|, \\ k_{G_2}(I'_2, K'_2) + \max\{1, h_{G_2}(I'_2, K'_2)/2\} &\leq |N_{G_2}(I'_2)| - |K'_2|. \end{aligned}$$

Since

$$\begin{aligned} &\max\{1, h_{G_1}(I'_1, K'_1)/2\} + \max\{1, h_{G_2}(I'_2, K'_2)/2\} \geq \\ &\max\{1, [h_{G_1}(I'_1, K'_1) + h_{G_2}(I'_2, K'_2)]/2\}, \end{aligned}$$

from the above inequalities we get $k_G(I', K') + \max\{1, h_G(I', K')/2\} \leq |N_G(I')| - |K'|$.

Now suppose that $(k_{G_1}(I'_1, K'_1), h_{G_1}(I'_1, K'_1)) = (0, 0)$. Then $(k_{G_2}(I'_2, K'_2), h_{G_2}(I'_2, K'_2)) \neq (0, 0)$. Since G_2 is a Burkard-Hammer graph, we have

$$\begin{aligned} &k_G(I', K') + \max\{1, h_G(I', K')/2\} \\ &= k_{G_2}(I'_2, K'_2) + \max\{1, h_{G_2}(I'_2, K'_2)/2\} \leq |N_{G_2}(I'_2)| - |K'_2| \\ &\leq |N_{G_2}(I'_2)| - |K'_2| + (|N_{G_1}(I'_1)| - |K'_1|) = |N_G(I')| - |K'|. \end{aligned}$$

A similar result can be obtained if $(k_{G_2}(I'_2, K'_2), h_{G_2}(I'_2, K'_2)) = (0, 0)$.

Thus, the Burkard-Hammer condition holds in Subcase (ii-1).

Subcase (ii-2): $v \in N_{G_1}(I'_1)$.
In this subcase, $K_2 \subseteq N_G(I')$. Therefore,

$$N_G(I') = K_2 \cup (N_{G_1}(I'_1) - v).$$

Let $K' \subseteq N_G(I')$. We again set $K'_1 = K' \cap (K_1 - v)$, $K'_2 = K' \cap K_2$, $K''_1 = K'_1 \cup \{v\}$. Then $K' = K'_1 \cup K'_2$ with $K'_1 \cap K'_2 = \emptyset$.

First suppose that $K'_2 = \emptyset$. Then $K' = K'_1$ and therefore $k_G(I', K') = k_{G_1}(I'_1, K'_1) + |I'_2|$ and $h_G(I', K') = h_{G_1}(I'_1, K'_1)$. If $(k_G(I', K'), h_G(I', K')) = (|I'_2|, 0) \neq (0, 0)$, then $k_G(I', K') + \max\{1, h_G(I', K')/2\} = |I'_2| + 1$. Meanwhile, $|N_G(I')| - |K'| \geq |K_2|$. Since $|I'_2| \leq |I_2| < |K_2|$, the Burkard-Hammer condition holds. If $(k_G(I', K'), h_G(I', K')) \neq (0, 0)$ and $\neq (|I'_2|, 0)$, then $(k_{G_1}(I'_1, K'_1), h_{G_1}(I'_1, K'_1)) \neq (0, 0)$. Therefore,

$$\begin{aligned} &k_G(I', K') + \max\{1, h_G(I', K')/2\} \\ &= k_{G_1}(I'_1, K'_1) + |I'_2| + \max\{1, h_{G_1}(I'_1, K'_1)/2\} \\ &\leq |N_{G_1}(I'_1)| - |K'_1| + |I'_2| \\ &\leq (|N_{G_1}(I'_1)| + |K_2| - 1) - |K'_1| = |N_G(I')| - |K'| \end{aligned}$$

and the Burkard-Hammer condition holds again.

Now suppose that $K'_2 \neq \emptyset$. Let A_1 be the component of $B_{G_1}(I'_1, K''_1)$ containing v. Since $K'_2 \neq \emptyset$ and G_2 is a complete split graph, it is not difficult to see that in $B_G(I', K')$ the induced subgraph on $[(V(A_1) - v) \cup I'_2 \cup K'_2]$, which we denote by A, is a component of $B_G(I', K')$. In this case,

$$\begin{aligned} I(A) &= I_1(A_1) \cup I'_2 \quad \text{with} \quad I_1(A_1) \cap I'_2 = \emptyset, \\ K(A) &= (K_1(A_1) - v) \cup K'_2 \quad \text{with} \quad (K_1(A_1) - v) \cap K'_2 = \emptyset. \end{aligned}$$

If $(k_G(I', K'), h_G(I', K')) \neq (0, 0)$ but $(k_{G_1}(I'_1, K''_1), h_{G_1}(I'_1, K''_1)) = (0, 0)$, then A_1 is an L-component. Hence, $|I_1(A_1)| < |K_1(A_1)|$. It follows that $|I_1(A_1)| \leq |K_1(A_1) - v|$. Since all components of $B_G(I', K')$ other than A are also components of $B_{G_1}(I'_1, K''_1)$, the component A has to be either an H-component or a T-component. In the former case, we have $|I(A)| = |K(A)|$. Therefore, $|I_1(A_1)| + |I'_2| = |I(A)| = |K(A)| = |K_1(A_1) - v| + |K'_2|$. Since $|I_1(A_1)| \leq |K_1(A_1) - v|$, it follows from the above equality that $|K'_2| \leq |I'_2| < |K_2|$. It is clear that in this case $k_G(I', K') = 0$ and $h_G(I', K') = 1$. It follows that $k_G(I', K') + \max\{1, h_G(I', K')/2\} = 1$. But

$$\begin{aligned} |N_G(I')| - |K'| &= (|K_2| + |N_{G_1}(I'_1) - v|) - (|K'_1| + |K'_2|) \\ &= (|K_2| - |K'_2|) + (|N_{G_1}(I'_1) - v| - |K'_1|)) \geq 1 \end{aligned}$$

and the Burkard-Hammer condition holds. In the latter case, i.e. when A is a T-component, we have $k_G(I', K') = |I(A)| - |K(A)|$ and $h_G(I', K') = 0$. Hence,

$$\begin{aligned}&k_G(I', K') + \max\{1, h_G(I', K')/2\} = |I(A)| - |K(A)| + 1\\ &= (|I_1(A_1)| + |I_2'|) - (|K_1(A_1) - v| + |K_2'|) + 1\\ &\le |I_2'| - |K_2'| + 1 \le (|K_2| - 1) - |K_2'| + 1 = |K_2| - |K_2'|.\end{aligned}$$

On the other hand, $|N_G(I')| - |K'| = (|N_{G_1}(I_1') - v| - |K_1'|) + (|K_2| - |K_2'|) \ge |K_2| - |K_2'|$. Hence, the Burkard-Hammer condition again holds.

Now assume $(k_G(I', K'), h_G(I', K')) \ne (0, 0)$ and $(k_{G_1}(I_1', K_1''), h_{G_1}(I_1', K_1''))$ $\ne (0, 0)$. In Column 2 of Table 1 we list all possibilities for types of A_1 in $B_{G_1}(I_1', K_1'')$ and for types of A in $B_G(I', K')$. The corresponding values of $k_G(I', K')$ and $h_G(I', K')$ are calculated in Column 3 of this table. We consider whether the Burkard-Hammer condition holds in each of this cases. In Case 1, we have

Table 1. The values of $k_G(I', K')$ and $h_G(I', K')$

No	**Types of $\mathbf{A_1}$ and $\mathbf{A}$**	**Values of $\mathbf{k_G(I', K')}$ and $\mathbf{h_G(I', K')}$**
1	Both A_1 and A are T-components.	$k_G(I', K') = k_{G_1}(I_1', K_1'') - k_{G_1}(A_1) + k_G(A)$, $h_G(I', K') = h_{G_1}(I_1', K_1'')$.
2	A_1 is a T-component, A is an H-component.	$k_G(I', K') = k_{G_1}(I_1', K_1'') - k_{G_1}(A_1)$, $h_G(I', K') = h_{G_1}(I_1', K_1'') + 1$.
3	A_1 is a T-component, A is an L-component.	$k_G(I', K') = k_{G_1}(I_1', K_1'') - k_{G_1}(A_1)$, $h_G(I', K') = h_{G_1}(I_1', K_1'')$.
4	A_1 is an H-component, A is a T-component.	$k_G(I', K') = k_{G_1}(I_1', K_1'') + k_G(A)$, $h_G(I', K') = h_{G_1}(I_1', K_1'') - 1$.
5	Both A_1 and A are H-components.	$k_G(I', K') = k_{G_1}(I_1', K_1'')$, $h_G(I', K') = h_{G_1}(I_1', K_1'')$.
6	A_1 is an H-component, A is an L-component.	$k_G(I', K') = k_{G_1}(I_1', K_1'')$, $h_G(I', K') = h_{G_1}(I_1', K_1'') - 1$.
7	A_1 is an L-component, A is an T-component.	$k_G(I', K') = k_{G_1}(I_1', K_1'') + k_G(A)$, $h_G(I', K') = h_{G_1}(I_1', K_1'')$.
8	A_1 is an L-component, A is an H-component.	$k_G(I', K') = k_{G_1}(I_1', K_1'')$, $h_G(I', K') = h_{G_1}(I_1', K_1'') + 1$.
9	Both A_1 and A are L-components.	$k_G(I', K') = k_{G_1}(I_1', K_1'')$, $h_G(I', K') = h_{G_1}(I_1', K_1'')$.

$$
\begin{aligned}
& k_G(I', K') + \max\{1, h_G(I', K')/2\} \\
&= k_{G_1}(I'_1, K''_1) - (|I_1(A_1)| - |K_1(A_1)|) + (|I(A)| - |K(A)|) + \\
&\quad \max\{1, h_{G_1}(I'_1, K''_1)/2\} \\
&= k_{G_1}(I'_1, K''_1) - |I_1(A_1)| + |K_1(A_1)| + |I_1(A_1)| + |I'_2| - \\
&\quad |K_1(A_1) - v| - |K'_2| + \max\{1, h_{G_1}(I'_1, K''_1)/2\} \\
&\le |N_{G_1}(I'_1)| - |K''_1| + |I'_2| - |K'_2| + 1 \\
&= |N_{G_1}(I'_1) - v| - |K'_1| + |I'_2| - |K'_2| + 1 \\
&\le (|N_{G_1}(I'_1) - v| + |K_2|) - (|K'_1| + |K'_2|) \\
&= |N_G(I')| - |K'|
\end{aligned}
$$

and the Burkard-Hammer condition holds.

In Cases 2 and 3 we have

$$
\begin{aligned}
& k_G(I', K') + \max\{1, h_G(I', K')/2\} \\
&\le k_{G_1}(I'_1, K''_1) - \ k_{G_1}(A_1) + \ \max\{1, \ [h_{G_1}(I'_1, \ K''_1) + 1]/2\} \\
&\le k_{G_1}(I'_1, \ K''_1) + \max\{1, \ h_{G_1}(I'_1, K''_1)/2\} \\
&\le |N_{G_1}(I'_1)| - |K''_1| \\
&\le |N_{G_1}(I'_1) - v| - |K'_1| + (|K_2| - |K'_2|) \\
&= |N_G(I')| - |K'|
\end{aligned}
$$

and the Burkard-Hammer condition also holds.

In Cases 4 and 7 we have

$$
\begin{aligned}
& k_G(I', K') + \max\{1, h_G(I', K')/2\} \\
&\le k_{G_1}(I'_1, K''_1) + (|I(A)| - |K(A)|) + \max\{1, h_{G_1}(I'_1, K''_1)/2\} \\
&\le |N_{G_1}(I'_1)| - |K''_1| + (|I_1(A_1)| + |I'_2|) - (|K_1(A_1) - v| + |K'_2|) \\
&= |N_{G_1}(I'_1) - v| - |K'_1| + (|I_1(A_1)| - |K_1(A_1)|) + 1 + |I'_2| - |K'_2| \\
&\le (|N_{G_1}(I'_1) - v| + |I'_2| + 1) - (|K'_1| + |K'_2|) \\
&\le (|N_{G_1}(I'_1) - v| + |K_2|) - (|K'_1| + |K'_2|) = |N_G(I')| - |K'|
\end{aligned}
$$

and the Burkard-Hammer condition holds.

In Cases 5, 6 and 9 we have

$$
\begin{aligned}
& k_G(I', K') + \max\{1, h_G(I', K')/2 \\
&\le k_{G_1}(I'_1, K''_1) + \max\{1, h_{G_1}(I'_1, K''_1)/2\} \\
&\le |N_{G_1}(I'_1)| - |K''_1| = |N_{G_1}(I'_1) - v| - |K'_1| \\
&\le |N_{G_1}(I'_1) - v| - |K'_1| + (|K_2| - |K'_2|) = |N_G(I')| - |K'|
\end{aligned}
$$

and the Burkard-Hammer condition holds.

In Case 8 we have

$$\begin{aligned} & k_G(I', K') + \max\{1, h_G(I', K')/2\} \\ &= k_{G_1}(I_1', K_1'') + \max\{1, [h_{G_1}(I_1', K_1'') + 1]/2\} \\ &\leq k_{G_1}(I_1', K_1'') + \max\{1, h_{G_1}(I_1', K_1'')/2\} + 1 \\ &\leq |N_{G_1}(I_1')| - |K_1''| + 1 \\ &= |N_{G_1}(I_1') - v| - |K_1'| + 1. \end{aligned}$$

In this case, $|I_1(A_1)| + |I_2'| = |I(A)| = |K(A)| = |K_1(A_1) - v| + |K_2'|$. Since A_1 is an L-component, we have $|I_1(A_1)| < |K_1(A_1)| \;\Leftrightarrow\; |I_1(A_1)| \leq |K_1(A_1) - v|$. It follows that $|K_2'| \leq |I_2'| \leq |K_2| - 1 \;\Leftrightarrow\; |K_2| - |K_2'| - 1 \geq 0$. Therefore,

$$\begin{aligned} & |N_{G_1}(I_1') - v| - |K_1'| + 1 \leq |N_{G_1}(I_1') - v| - |K_1'| + 1 + (|K_2| - |K_2'| - 1) \\ &= (|N_{G_1}(I_1') - v| + |K_2|) - (|K_1'| + |K_2'|) = |N_G(I')| - |K'| \end{aligned}$$

and the Burkard-Hammer condition holds.

The proof of Theorem 2 is complete.

Theorem 3. Let $G_1 = S(I_1 \cup K_1, E_1)$ be an arbitrary split graph and $G_2 = S(I_2 \cup K_2, E_2)$ be a split graph with $|K_2| = |I_2| + 1$. Then an expansion of G_1 by G_2 is a hamiltonian graph if and only if both G_1 and G_2 are hamiltonian graphs.

Proof. Let $v \in K_1$ and $G = G_1[G_2, v]$ be an expansion of G_1 by G_2 at v.

First suppose that G has a Hamilton cycle C. By $\overrightarrow{C}$ we denote the cycle C with a given orientation and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices in the reverse order are given by $v\overleftarrow{C}u$. We will consider $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. If $u \in V(C)$, then u^+ denotes the successor of u on $\overrightarrow{C}$, and u^- denotes its predecessor. Similar notations as described above for cycles are used for paths.

Consider in $\overrightarrow{C}$ the paths $P_1 = x_1\overrightarrow{C}y_1$, $P_2 = x_2\overrightarrow{C}y_2$,..., $P_t = x_t\overrightarrow{C}y_t$ with the following properties:

(a) Vertices of I_2 and K_2 occur alternatively in P_i, $i = 1, 2, \ldots, t$;
(b) The endvertices x_i and y_i of each $P_i, i = 1, 2, \ldots, t$, are in K_2;
(c) x_i^- and y_i^+ are not in I_2 for every $i = 1, 2, \ldots, t$;
(d) Every vertex of I_2 is in one of $P_1, P_2, \ldots, P_t$.

Let ℓ_i be the number of vertices of I_2 in P_i. Then it is clear that the number of vertices of K_2 in P_i is $\ell_i + 1$. By (d), $\ell_1 + \ell_2 + \cdots + \ell_t = |I_2|$. So in total, the number of vertices of K_2 in all paths $P_1, P_2, \ldots, P_t$ is $|I_2| + t$. It follows that $|I_2| + t \leq |K_2| \;\Leftrightarrow\; |I_2| + t \leq |I_2| + 1 \;\Leftrightarrow\; t \leq 1$ and therefore $t = 1$.

Thus, $\overrightarrow{C}$ contains the path $x_1\overrightarrow{C}y_1$ which is a Hamilton path of G_2. Since both x_1 and y_1 are in K_2, we have $x_1y_1 \in E_2$ and therefore $C_2 = x_1\overrightarrow{P}y_1x_1$ is a Hamilton cycle of G_2. Further, by the definition of $G_1[G_2, v]$, both x_1^- and y_1^+ are adjacent in G_1 to v. So $C_1 = vx_1^-\overleftarrow{C}y_1^+v$ is a Hamilton cycle of G_1. The necessity is proved.

Now suppose that G_1 has a Hamilton cycle with a given orientation $\overrightarrow{C}_1$ and G_2 has a Hamilton cycle with a given orientation $\overrightarrow{C}_2$. Since $|K_2| > |I_2|$, there are vertices $v_2 \in K_2$ such that $v_2^+ \in K_2$. Then $C = v^+\overrightarrow{C}_1 v^- v_2^+ \overrightarrow{C}_2 v_2 v^+$ is a Hamilton cycle of G. Thus, the sufficiency is proved.

The proof of Theorem 3 is complete.

From Theorems 2 and 3 we obtain immediately that any expansion of the graphs H^1, H^2, H^3 and H^4, defined in Section 2, by a complete split graph $G_2 = S(I_2 \cup K_2, E_2)$ with $|K_2| = |I_2| + 1$ is a nonhamiltonian Burkard-Hammer graph. This result is a particular case of our more general Theorem 5. If now we apply Theorems 2 and 3 to nonhamiltonian Burkard-Hammer graphs just obtained, then we get further new nonhamiltonian Burkard-Hammer graphs. By repeating this procedure we can get other new ones. Thus, by Theorems 2 and 3, different kinds of nonhamiltonian Burkard-Hammer graphs can be obtained recursively from H^1, H^2, H^3 and H^4.

4 Construction 2

Let H^1 be the graph defined in Section 2. For $t = 0, 1, 2, \ldots$ we define the graphs $H_t^1 = S(I_t \cup K_t, E_t)$ from H^1 recursively as follows: $H_0^1 = H^1$ and if $H_t^1 = S(I_t \cup K_t, E_t)$ has been defined, then $H_{t+1}^1 = S(I_{t+1} \cup K_{t+1}, E_{t+1})$, where $I_0 = I(H^1), K_0 = K(H^1), E_0 = E(H^1)$, $v_0^* = v_1 \in K(H^1)$ and

$$\begin{aligned} I_{t+1} &= I_t \cup \{u_{t+1}^*\}, \\ K_{t+1} &= K_t \cup \{v_{t+1}^*\}, \\ E_{t+1} &= E_t \cup \{u_{t+1}^* v_{t+1}^*, u_{t+1}^* v_t^*, u_4 v_{t+1}^*, v_{t+1}^* v \mid v \in K_t\}. \end{aligned}$$

We illustrate the graph H_3^1 in Figure 3. Further, for $t = 0, 1, 2, \ldots$ we define
$H_t^2 = H_t^1 + u_4 v_2$,
$H_t^3 = H_t^1 + u_5 v_2$,
$H_t^4 = H_t^1 + \{u_4 v_2, u_5 v_2\}$.

Theorem 4. For any $t = 0, 1, 2, \ldots$, the graphs H_t^1, H_t^2, H_t^3 and H_t^4 are nonhamiltonian Burkard-Hammer graphs.

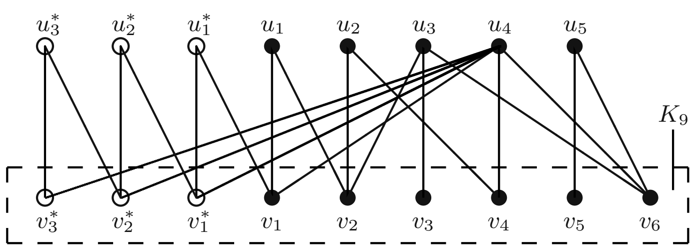

Fig. 3. The graph H_3^1

Proof. First we prove the assertion for H_t^1 by induction on t.

For $t = 0$, the assertion is true because H^1 is known to be a nonhamiltonian Burkard-Hammer graph.

Suppose now that H_t^1 has been proved to be a nonhamiltonian Burkard-Hammer graph. If H_{t+1}^1 has a Hamilton cycle C, then both edges $u_{t+1}^*v_{t+1}^*$ and $u_{t+1}^*v_t^*$ are in C because $\deg(u_{t+1}^*) = 2$. We fix an orientation on C so that $(u_{t+1}^*)^+ = v_t^*$ and $(u_{t+1}^*)^- = v_{t+1}^*$. If $(v_{t+1}^*)^- \in K_t$, then $(v_{t+1}^*)^-$ is adjacent to v_t^* in H_t^1. Therefore, $C' = v_t^*\overrightarrow{C}(v_{t+1}^*)^-v_t^*$ is a Hamilton cycle of H_t^1, a contradiction. If $(v_{t+1}^*)^- \notin K_t$, then $(v_{t+1}^*)^- = u_4$ by the definition of H_{t+1}^1. But u_4 is adjacent to v_t^* in H_t^1. So $C' = v_t^*\overrightarrow{C}v_4v_t^*$ is a Hamilton cycle of H_t^1, a contradiction again. Thus, H_{t+1}^1 is nonhamiltonian.

Now we prove that H_{t+1}^1 is a Burkard-Hammer graph. Let $\emptyset \neq I' \subseteq I_{k+1}$, $K' \subseteq N_{H_{t+1}^1}(I'), I_t' = I' \cap I_t$ and $K_t' = K' \cap K_t$. We consider separately the following cases.

Case (i): $I' \subseteq I_t$.
If $K' \subseteq K_t$, then $B_{H_{t+1}^1}(I',K') = B_{H_t^1}(I',K')$. Therefore, $k_{H_{t+1}^1}(I',K')$ $= k_{H_t^1}(I',K')$ and $h_{H_{t+1}^1}(I',K') = h_{H_t^1}(I',K')$. It follows that if $(k_{H_{t+1}^1}(I', K'),$ $h_{H_{t+1}^1}(I',K')) \neq (0,0)$, then $(k_{H_t^1}(I',K'), h_{H_t^1}(I',K')) \neq (0,0)$ and

$$\begin{aligned}
&k_{H_{t+1}^1}(I',K') + \max\{1, h_{H_{t+1}^1}(I',K')/2\}\\
&= k_{H_t^1}(I',K') + \max\{1, h_{H_t^1}(I',K')/2\}\\
&\leq |N_{H_t^1}(I')| - |K'| \leq |N_{H_{t+1}^1}(I')| - |K'|.
\end{aligned}$$

Thus, the Burkard-Hammer condition holds in this subcase.

If $K' \nsubseteq K_t$, then K' contains v_{t+1}^* and therefore I' has to contain u_4. Let A_t and A_{t+1} be the components of $B_{H_t^1}(I',K_t')$ and $B_{H_{t+1}^1}(I',K')$ which contain the vertex u_4, respectively. Then it is not difficult to see that A_{t+1} can be obtained from A_t by adding the vertex v_{t+1}^* and the edge $u_4v_{t+1}^*$. Therefore, if $(k_{H_{t+1}^1}(I',K'), h_{H_{t+1}^1}(I',K')) \neq (0,0)$, then $(k_{H_t^1}(I',K_t'), h_{H_t^1}(I',K_t')) \neq (0,0)$ and

$$\begin{aligned}
&k_{H_{t+1}^1}(I',K') + \max\{1, h_{H_{t+1}^1}(I',K')/2\}\\
&\leq k_{H_t^1}(I',K_t') + \max\{1, h_{H_t^1}(I',K_t')/2\}\\
&\leq |N_{H_t^1}(I')| - |K_t'| = (|N_{H_t^1}(I')| + 1) - (|K_t'| + 1)\\
&= |N_{H_{t+1}^1}(I')| - |K'|.
\end{aligned}$$

So the Burkard-Hammer condition again holds in this subcase.

Case (ii): $I' \nsubseteq I_t$.
In this case, $u_{t+1}^* \in I'$ and $v_{t+1}^*, v_t^* \in N_{H_{t+1}^1}(I')$. We again divide this case into several subcases.

Subcase (ii-1): K' contains neither v^*_{t+1} nor v^*_t.

In this subcase, $A_{t+1} = \{u^*_{t+1}\}$ constitutes a T-component of $B_{H^1_{t+1}}(I', K')$ with $k_{H^1_{t+1}}(A_{t+1}) = 1$ and $k_{H^1_{t+1}}(I', K') = k_{H^1_t}(I'_t, K'_t) + 1$, $h_{H^1_{t+1}}(I', K') = h_{H^1_t}(I'_t, K'_t)$.

If $(k_{H^1_{t+1}}(I', K'), h_{H^1_{t+1}}(I', K')) = (1, 0)$, then the Burkard-Hammer condition trivially holds because $v^*_{t+1}, v^*_t \in N_{H^1_{t+1}}(I')$ but they are not in K'. So we may assume $(k_{H^1_{t+1}}(I', K'), h_{H^1_{t+1}}(I', K')) \neq (0, 0)$ and $\neq (1, 0)$. Then $(k_{H^1_t}(I'_t, K'_t), h_{H^1_t}(I'_t, K'_t)) \neq (0, 0)$. Therefore,

$$
\begin{aligned}
&k_{H^1_{t+1}}(I', K') + \max\{1, h_{H^1_{t+1}}(I', K')/2\} \\
&= (k_{H^1_t}(I'_t, K'_t) + 1) + \max\{1, h_{H^1_t}(I'_t, K'_t)/2\} \\
&\leq |N_{H^1_t}(I'_t)| + 1 - |K'_t| = |N_{H^1_{t+1}}(I')| - |K'|.
\end{aligned}
$$

Thus, the Burkard-Hammer condition holds in this subcase.

Subcase (ii-2): K' contains v^*_t.

Let A_t and A_{t+1} be the components of $B_{H^1_t}(I'_t, K'_t)$ and $B_{H^1_{t+1}}(I', K')$, which contain the vertex v^*_t, respectively.

If K' does not contain the vertex v^*_{t+1}, then $I(A_{t+1}) = I(A_t) \cup \{u^*_{t+1}\}$ and $K(A_{t+1}) = K(A_t)$. Therefore, in order to have $(k_{H^1_{t+1}}(I', K'), h_{H^1_{t+1}}(I', K'))$ $\neq (0, 0)$, but $(k_{H^1_t}(I'_t, K'_t), h_{H^1_t}(I'_t, K'_t)) = (0, 0)$, the component A_{t+1} must be an H-component and all other components of $B_{H^1_{t+1}}(I', K')$ are L-components. It follows that $k_{H^1_{t+1}}(I', K') = 0, h_{H^1_{t+1}}(I', K') = 1$ and the Burkard-Hammer condition holds because $v^*_{t+1} \in N_{H^1_{t+1}}(I')$ but $v^*_{t+1} \notin K'$. So we may assume that if $(k_{H^1_{t+1}}(I', K'), h_{H^1_{t+1}}(I', K')) \neq (0, 0)$, then $(k_{H^1_t}(I'_t, K'_t), h_{H^1_t}(I'_t, K'_t)) \neq (0, 0)$. In this situation, we have

$$
\begin{aligned}
&k_{H^1_{t+1}}(I', K') + \max\{1, h_{H^1_{t+1}}(I', K')/2\} \\
&\leq k_{H^1_t}(I'_t, K'_t) + \max\{1, h_{H^1_t}(I'_t, K'_t)/2\} + 1 \\
&\leq |N_{H^1_t}(I'_t)| - |K'_t| + 1 = |N_{H^1_{t+1}}(I')| - |K'|
\end{aligned}
$$

and the Burkard-Hammer condition holds.

If K' contains v^*_{t+1}, then $I(A_{t+1}) = I(A_t) \cup \{u^*_{t+1}\}$ and $K(A_{t+1}) = K(A_t) \cup \{v^*_{t+1}\}$. Therefore, $k_{H^1_{t+1}}(I', K') = k_{H^1_t}(I'_t, K'_t)$, $h_{H^1_{t+1}}(I', K') = h_{H^1_t}(I'_t, K'_t)$. So if $(k_{H^1_{t+1}}(I', K'), h_{H^1_{t+1}}(I', K')) \neq (0, 0)$, then $(k_{H^1_t}(I'_t, K'_t), h_{H^1_t}(I'_t, K'_t)) \neq (0, 0)$ and

$$
\begin{aligned}
&k_{H^1_{t+1}}(I', K') + \max\{1, h_{H^1_{t+1}}(I', K')/2\} \\
&= k_{H^1_t}(I'_t, K'_t) + \max\{1, h_{H^1_t}(I'_t, K'_t)/2\} \\
&\leq |N_{H^1_t}(I'_t)| - |K'_t| = (|N_{H^1_t}(I'_t)| + 1) - (|K'_t| + 1) \\
&= |N_{H^1_{t+1}}(I')| - |K'|
\end{aligned}
$$

and again the Burkard-Hammer condition holds.

Subcase (ii-3): K' does not contain v_t^* but contains v_{t+1}^*.

First assume that $u_4 \in I'$. Let A_t and A_{t+1} be the components of $B_{H_t^1}(I_t', K_t')$ and $B_{H_{t+1}^1}(I', K')$, which contain the vertex u_4, respectively. Then again $I(A_{t+1}) = I(A_t) \cup \{u_{t+1}^*\}$ and $K(A_{t+1}) = K(A_t) \cup \{v_{t+1}^*\}$. By calculations similar to those of Subcase (ii-2) (for K' containing v_{t+1}^*) we see that the Burkard-Hammer condition holds.

Now suppose that $u_4 \notin I'$. Then $B_{H_{t+1}^1}(I', K')$ has an H-component T with $I(T) = \{u_{t+1}^*\}$ and $K(T) = \{v_{t+1}^*\}$. All other components of $B_{H_{t+1}^1}(I', K')$ are also components of $B_{H_t^1}(I_t', K_t')$. Therefore, $k_{H_{t+1}^1}(I', K') = k_{H_t^1}(I_t', K_t')$ and $h_{H_{t+1}^1}(I', K') = h_{H_t^1}(I_t', K_t') + 1$.

If $(k_{H_{t+1}^1}(I', K'), h_{H_{t+1}^1}(I', K')) = (0, 1)$, then it is not difficult to see that the Burkard-Hammer condition holds in this situation because $v_t^* \in N_{H_{t+1}^1}(I')$ but $v_t^* \notin K'$. So we may assume that $(k_{H_{t+1}^1}(I', K'), h_{H_{t+1}^1}(I', K')) \neq (0, 0)$ and $\neq (0, 1)$. Then $(k_{H_t^1}(I_t', K_t'), h_{H_t^1}(I_t', K_t')) \neq (0, 0)$ and therefore the Burkard-Hammer condition holds for these I_t' and K_t'. If $u_t^* \notin I'$, then $u_t^* \notin I_t'$. Therefore, both v_{t+1}^* and v_t^* are not in $N_{H_t^1}(I_t')$. It follows that $|N_{H_{t+1}^1}(I')| = |N_{H_t^1}(I_t')| + 2$. Hence,

$$\begin{aligned}
&k_{H_{t+1}^1}(I', K') + \max\{1, h_{H_{t+1}^1}(I', K')/2\} \\
&= k_{H_t^1}(I_t', K_t') + \max\{1, [h_{H_t^1}(I_t', K_t') + 1]/2\} \\
&\leq k_{H_t^1}(I_t', K_t') + \max\{1, h_{H_t^1}(I_t', K_t')/2\} + 1 \\
&\leq |N_{H_t^1}(I_t')| - |K_t'| + 1 = (|N_{H_t^1}(I_t')| + 2) - (|K_t'| + 1) \\
&= |N_{H_{t+1}^1}(I')| - |K'|,
\end{aligned}$$

and the Burkard-Hammer condition holds in this situation. So we may assume further that $u_t^* \in I'$.

Set $K_t'' = K_t' \cup \{v_t^*\}$ and consider $B_{H_t^1}(I_t', K_t'')$. Let B_t and A_t be the components of $B_{H_t^1}(I_t', K_t'')$ and $B_{H_t^1}(I_t', K_t')$, which contain u_t^*, respectively. Then $A_t = B_t - v_t^*$.

If $(k_{H_{t+1}^1}(I', K'), h_{H_{t+1}^1}(I', K')) = (0, 1)$ or $(0, 2)$, then the Burkard-Hammer condition holds because $v_t^* \in N_{H_{t+1}^1}(I')$ but $v_t^* \notin K'$. So we may assume further that $(k_{H_{t+1}^1}(I', K'), h_{H_{t+1}^1}(I', K'))$ is different than $(0, 0), (0, 1)$ and $(0, 2)$. Then $(k_{H_t^1}(I_t', K_t'), h_{H_t^1}(I_t', K_t'))$ is different than $(0, 0)$ and $(0, 1)$. It follows that $(k_{H_t^1}(I_t', K_t''), h_{H_t^1}(I_t', K_t'')) \neq (0, 0)$. Since H_t^1 is a Burkard-Hammer graph, we have

$$\begin{aligned}
&k_{H_t^1}(I_t', K_t'') + \max\{1, h_{H_t^1}(I_t', K_t'')/2\} \\
&\leq |N_{H_t^1}(I_t')| - |K_t''| = |N_{H_t^1}(I_t')| - |K_t'| - 1 \\
&< |N_{H_t^1}(I_t')| - |K_t'|.
\end{aligned}$$

If B_t is an L-component, then A_t is either an L-component or an H-component. So $k_{H_t^1}(I_t', K_t') = k_{H_t^1}(I_t', K_t'')$ and $h_{H_t^1}(I_t', K_t') \leq h_{H_t^1}(I_t', K_t'') + 1$. Therefore,

$$\begin{aligned}
&k_{H^1_{t+1}}(I',K') + \max\{1, h_{H^1_{t+1}}(I',K')/2\}\\
&= k_{H^1_t}(I'_t,K'_t) + \max\{1, [h_{H^1_t}(I'_t,K'_t)+1]/2\}\\
&\le k_{H^1_t}(I'_t,K''_t) + \max\{1, h_{H^1_t}(I'_t,K''_t)/2\} + 1\\
&\le |N_{H^1_t}(I'_t)| - |K''_t| + 1 = |N_{H^1_t}(I'_t)| - |K'_t|\\
&= |N_{H^1_{t+1}}(I')| - |K'|
\end{aligned}$$

and the Burkard-Hammer condition holds.

If B_t is an H-component, then A_t is a T-component with $k_{H^1_t}(A_t) = 1$. So $k_{H^1_t}(I'_t,K'_t) = k_{H^1_t}(I'_t,K''_t) + 1$ and $h_{H^1_t}(I'_t,K'_t) = h_{H^1_t}(I'_t,K''_t) - 1$. Therefore,

$$\begin{aligned}
&k_{H^1_{t+1}}(I',K') + \max\{1, h_{H^1_{t+1}}(I',K')/2\}\\
&= k_{H^1_t}(I'_t,K'_t) + \max\{1, [h_{H^1_t}(I'_t,K'_t)+1]/2\}\\
&= k_{H^1_t}(I'_t,K''_t) + 1 + \max\{1, h_{H^1_t}(I'_t,K''_t)/2\}\\
&\le |N_{H^1_t}(I'_t)| - |K''_t| + 1 = |N_{H^1_t}(I'_t)| - |K'_t|\\
&= |N_{H^1_{t+1}}(I')| - |K'|
\end{aligned}$$

and the Burkard-Hammer condition holds again.

Finally, suppose that B_t is a T-component. Then since $u_4 \notin I'$, it is clear that all vertices $u^*_t, \ldots, u^*_1, u_1, u_2, u_3$ are in $I(B_t)$ and all vertices $v^*_t, \ldots, v^*_1, v_1, v_2$ are in $K(B_t)$. So either $I'_t = \{u^*_t, \ldots, u^*_1, u_1, u_2, u_3\}$ or $I'_t = \{u^*_t, \ldots, u^*_1, u_1, u_2, u_3, u_5\}$. In the former case, $K'_t = \{v^*_{t-1}, \ldots, v^*_1, v_1, v_2\}$. Therefore, $k_{H^1_{t+1}}(I',K') = 2$ and $h_{H^1_{t+1}}(I',K') = 1$. It follows that

$$k_{H^1_{t+1}}(I',K') + \max\{1, h_{H^1_{t+1}}(I',K')/2\} = 3.$$

On the other hand, $|N_{H^1_{t+1}}(I')| = |\{v^*_{t+1}, \ldots, v^*_1, v_1, v_2, v_3, v_4, v_6\}| = t+6$ and $|K'| = |\{v^*_{t+1}, v^*_{t-1}, \ldots, v^*_1, v_1, v_2\}| = t+2$. So the Burkard-Hammer condition holds. In the latter case, either $I(B_t) = \{u^*_t, \ldots, u^*_1, u_1, u_2, u_3\}, K(B_t) = \{v^*_t, \ldots, v^*_1, v_1, v_2\}$ or $I(B_t) = \{u^*_t, \ldots, u^*_1, u_1, u_2, u_3, u_5\}$ and $K(B_t) = \{v^*_t, \ldots, v^*_1, v_1, v_2, v_6\}$. If $I(B_t) = \{u^*_t, \ldots, u^*_1, u_1, u_2, u_3\}$ and $K(B_t) = \{v^*_t, \ldots, v^*_1, v_1, v_2\}$, then the component C of $B_{H^1_t}(I'_t, K''_t)$ containing u_5 either has $I(C) = \{u_5\}$ and $K(C) = \emptyset$ or has $I(C) = \{u_5\}$ and $K(C) = \{v_5\}$. If $I(B_t) = \{u^*_t, \ldots, u^*_1, u_1, u_2, u_3, u_5\}$, then B_t is the only component of $B_{H^1_t}(I'_t, K''_t)$. It is not difficult to see that in any of these situations, the Burkard-Hammer condition holds for H^1_{t+1}.

Thus, we have finished the proof that H^1_t is a nonhamiltonian Burkard-Hammer graph for any $t = 0, 1, 2, \ldots$.

Now consider H^2_t, H^3_t and H^4_t. Since H^1_t is a Burkard-Hammer graph, the graphs H^2_t, H^3_t and H^4_t are also Burkard-Hammer graphs. The proof of nonhamiltonicity of H^2_t, H^3_t and H^4_t are similar to that of H^1_t.

The proof of Theorem 4 is complete.

From Theorems 2, 3 and 4 we obtain immediately the following theorem.

Theorem 5. Any expansion of H^1_t, H^2_t, H^3_t and H^4_t $(t = 0, 1, 2 \ldots)$ by a complete split graph $G_2 = S(I_2 \cup K_2, E_2)$ with $|K_2| = |I_2| + 1$ is a nonhamiltonian Burkard-Hammer graph.

The following corollary is a particular case of Theorem 5, but it has some own interest by providing an infinite family of nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with the minimum degree $\delta(G) \geq |I| - 4$. We recall that it is already known that there are no nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with the minimum degree $\delta(G) \geq |I| - 2$ and there are only finitely many nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with the minimum degree $\delta(G) \geq |I| - 3$, namely the graphs H^1, H^2, H^3 and H^4.

Theorem 6. Any expansion $G = S(I \cup K, E)$ of H^4 by a complete split graph $G_2 = S(I_2 \cup K_2, E_2)$ with $|K_2| = |I_2| + 1$ at the vertex v_2 is a nonhamiltonian Burkard-Hammer graph with the minimum degree $\delta(G) \geq |I| - 4$.

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A Characterization of Polygonal Regions Searchable from the Boundary

Xuehou Tan

Tokai University, 317 Nishino, Numazu 410-0395, Japan
tan@wing.ncc.u-tokai.ac.jp

Abstract. We consider the problem of searching for a moving target with unbounded speed in a dark polygonal region by a searcher. The searcher continuously moves on the polygon boundary and can see only along the rays of the flashlights emanating from his position at a time. We present necessary and sufficient conditions for a polygon of n vertices to be *searchable from the boundary*. Our two main results are the following:

1. We present an $O(n \log n)$ time and $O(n)$ space algorithm for testing the searchability of simple polygons. Moreover, a search schedule can be reported in time linear in its size I, if it exists. For the searcher having full 360° vision, $I < 2n$, and for the searcher having only one flashlight, $I < 3n^2$. Our result improves upon the previous $O(n^2)$ time and space solution, given by LaValle et al [5]. Also, the linear bound for the searcher having full 360° vision solves an open problem posed by Suzuki et al [7].
2. We show the equivalence of the abilities of the searcher having only one flashlight and the one having full 360° vision. Although the same result has been obtained by Suzuki et al [7], their proof is long and complicated, due to lack of the characterization of boundary search.

1 Introduction

Recently, much attention has been devoted to the problem of searching for an unpredictable, moving target with unbounded speed in an n-sided polygon P by a mobile searcher [5, 6, 7]. Both the searcher and the target are modeled by points that can continuously move in P. A searcher is called the *k-searcher* if he holds k flashlights, and can see only along the rays of the flashlights emanating from his position at a time, or the *∞-searcher* if he has a light bulb that gives full 360° vision. The searcher can rotate a flashlight, with bounded speed to change the direction of the flashlight. The objective is to decide whether there exists a *search schedule* for the searcher to detect the target (i.e., the target is finally illuminated by the ray of some flashlight, no matter how he moves), and if so, generate a search schedule. A polygon is said to be *k-searchable* or *∞-searchable* if there exists a search schedule for the searcher to detect the target.

Motivated by robotics applications, LaValle *et al.* considered a simple model, in which the searcher continuously moves on the boundary of P and holds only

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one flashlight [5]. By constructing a two-dimensional diagram of size $\Omega(n^2)$, they gave an $O(n^2)$ time and space algorithm for generating a search schedule, if it exists [5]. On the other hand, Suzuki *et al.* showed that any polygon searchable by the ∞-searcher from the boundary is also searchable by the 1-searcher from the boundary [7]. Due to lack of the characterization of boundary search, their proof is long and complicated. Whether or not a good (e.g., linear) bound on the size of search schedules for the ∞-searcher can be established is left as an open problem in [7].

In this paper, we present necessary and sufficient conditions for a polygon to be *searchable from the boundary*, and provide efficient algorithms for determining the searchability of simple polygons and generating a search schedule if it exists. The first necessary condition states that a polygon P is not searchable from the boundary if there are three points p_1, p_2, p_3 on the boundary of P such that the Euclidean shortest path between any pair of p_i, p_j $(i, j \in \{1, 2, 3\})$ within P contains no point visible from the third point p_k ($k \neq i$ or j). The second and third conditions together state that a polygon P is not searchable from the boundary if *every* boundary point of P is surrounded by at least one of three special configurations; these configurations provide a place for the target to defend himself from the first (or initial) attack made by the searcher. If none of these conditions is true, then P is searchable from the boundary.

The paper is structed as follows. Section 2 reviews the two-guard problem [3, 4], which is used as a subroutine in our search algorithm. In Section 3, we give three necessary and sufficient conditions for the polygons to be searchable from the boundary. Based on this characterization, the equivalence of the abilities of the 1-searcher and the ∞-searcher is established. In Section 4, we describe an $O(n \log n)$ and $O(n)$ space algorithm for testing the searchability of simple polygons. A search schedule can be reported in time linear in its size I, if it exists. For the ∞-searcher, $I < 2n$, and for the 1-searcher, $I < 3n^2$.

2 Review of the Two-Guard Problem

Let P denote a simple polygon (without holes or self-intersections). Two points $x, y \in P$ are said to be mutually *visible* if the segment $\overline{xy}$ is entirely contained in P. For two regions $P_1, P_2 \subseteq P$, we say that P_1 is *weakly visible* from P_2 if every point in P_1 is visible from some point in P_2.

A *corridor* P is a simple polygon with two marked boundary points u and v. The two-guard problem [3, 4] asks if there exists a walk such that two guards l and r move along two polygonal chains L and R oriented from u to v, one clockwise and one counterclockwise, in such a way that l and r are always mutually visible. For two points $p, p' \in L$, we say that p *precedes* p' (and p' *succeeds* p) if we encounter p before p' when traversing L from u to v. If p precedes p', we write $p < p'$. We define these concepts for R in a similar manner.

For a vertex x of a polygonal chain, let $Succ(x)$ denote the vertex of the chain immediately succeeding x, and $Pred(x)$ the vertex immediately preceding x. A vertex of P is *reflex* if its internal angle is strictly larger than π. The backward ray shot from a reflex vertex r, denoted by *Backw(r)*, is the first boundary point of P hit by a "bullet" shot at r in the direction from $Succ(r)$ to r, and the forward ray shot *Forw(r)* is the first point hit by the bullet shot at r in the direction from $Pred(r)$ to r (Fig. 1). A pair of reflex vertices $p \in L$, $q \in R$ is said to give a *deadlock* if $q < Backw(p) \in R$ and $p < Backw(q) \in L$ hold or if $q > Forw(p) \in R$ and $p > Forw(q) \in L$ hold. See Fig. 1.

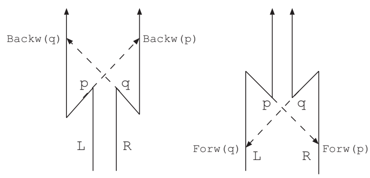

Fig. 1. Deadlocks

Lemma 1. [4] A corridor P is walkable if and only if the chains L and R are mutually weakly visible, and no deadlocks occur.

Also, a walk from one segment $\overline{p_0q_0}$ to another segment $\overline{p_1q_1}$, where $p_0 < p_1$ and $q_0 < q_1$, is possible if and only if two subchains from p_0 to p_1 and from q_0 to q_1 are mutually weakly visible and no deadlocks occur between them. For a walkable corridor, we need to give a *walk schedule*. The walk schedule consists of the following elementary actions: (i) both guards move forward along single edges, and (ii) one guard moves forward, but the other moves backward, along segments of single edges.

Lemma 2. [3, 4] It takes $O(n)$ time to test the two-guard walkability of a corridor, and $O(nlogn + I)$ time to generate a walk schedule, where I $(\leq n^2)$ is the minimal number of walk instructions.

3 Searching a Polygon from the Boundary

Let P be a simple polygon. Given a boundary point d, we can order all boundary points counterclockwise, starting and ending at d. For a complete ordering, we consider d as two points d_l and d_r such that $d_l \leq p \leq d_r$, for all points p on the boundary of P. Similar definitions can then be given as those in Section 2. For two boundary points p, p', we say that p *precedes* p' (and p' *succeeds* p) if we encounter p before p' when traversing from d_l to d_r. We write $p < p'$ if p precedes p'. For a vertex x, we denote by $Succ(x)$ the vertex immediately succeeding x, and $Pred(x)$ the vertex immediately preceding x. For a reflex vertex r, the backward and forward ray shots $Backw(r)$ and $Forw(r)$ are the first boundary points of P hit by the bullets shot at r in the directions from $Succ(r)$ to r and from $Pred(r)$ to r, respectively. In the case that d is a reflex vertex, the shots $Backw(d)$ and $Forw(d)$ can similarly be defined using $Succ(d_l)$

and $Pred(d_r)$. A pair of reflex vertices x, y is said to give a *deadlock* for the point p if $p_l < x < Forw(y) < Backw(x) < y < p_r$ holds.

In order to simplify the presentation, we denote, by $[u, v]$, the boundary interval from u to v counterclockwise. For an interval X, the point $y \in X$ is said to be the *maximum* (resp. *minimum*), if $y \geq x \in X$ (resp. $y \leq x \in X$).

3.1 Necessity

We present three necessary conditions for a polygon P to be searchable by the ∞-searcher from the boundary. A point $x \in P$ is said to be *detected* or *illuminated* at a time t, if x is contained in the region that is visible from the position of the ∞-searcher at t. Any region that might contain the target at a time is said to be *contaminated*; otherwise, it is said to be *clear*. If a region becomes contaminated for the second or more time, it is referred to as *recontaminated.*

What important in clearing P is to avoid a 'cycle' of recontaminations. Obviously, a cycle of recontanimations occurs if there are three boundary points such that when the ∞-searcher moves between any two of them, the third point is contaminated or recontaminated (Fig. 2).

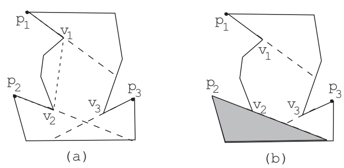

Fig. 2. A polygon satisfying the condition **C1**

Theorem 1. A simple polygon is not ∞-searchable from the boundary if (**C1**) there are three points p_1, p_2 and p_3 on the boundary such that the shortest path between any pair of p_i, p_j $(i, j \in \{1, 2, 3\})$ within the polygon contains no point visible from the third point p_k $(k \neq i$ or $j)$.

Proof. Assume that P is a simple polygon. Let p_1, p_2 and p_3, given in counterclockwise order, denote three boundary points of P which satisfy the condition **C1**. See Fig. 2. Without loss of generality, assume that the ∞-searcher starts at p_1. To clear the next point, say, p_2, it suffices for the ∞-searcher to move within the interval $[p_1, p_2]$ (Fig. 2a). Since the shortest path between p_1 and p_2 contains no point visible from p_3, the third point p_3 remains contaminated when the ∞-searcher moves within $[p_1, p_2]$. To clear the point p_3, the ∞-searcher has to move outside of the interval $[p_1, p_2]$ at least once. However, when the ∞-searcher moves to the point p_2 (resp. p_1), the target may sneak from p_3 into p_1 (resp. p_2). See Fig. 2b for an example, where the ∞-searcher is located at the point p_2 and the cleared region is shaded. Thus, whenever the ∞-searcher moves within $[p_i, p_j]$ $(i, j \in \{1, 2, 3\})$, the third point p_k $(k \neq i$ or $j)$ is contaminated or recontaminated. Hence, P is not searchable from the boundary. $\square$

For any three points p_1, p_2 and p_3 satisfying **C1**, we can find the reflex vertices v_1, v_2 and v_3 such that three adjacent vertices of them, each per vertex v_i ($1 \le i \le 3$), satisfy the condition **C1**. See Fig. 2a for an example, where p_i is just the vertex adjacent to v_i. The condition **C1** is then said to become true due to the existence of v_1, v_2 and v_3, or shortly, due to v_1, v_2 and v_3.

There are some other cases in which a cycle of recontaminations occurs (Fig. 3). We need more definitions. A pair of vertices v_1, v_2 is said to give a *BF-pair* for a boundary point p if $p_l < v_1 < Backw(v_1) < v_2 < p_r$ and $v_1 < Forw(v_2)$ hold. For the polygon shown in Fig. 3a, each boundary point has a BF-pair. A triple of vertices v_1, v_2 and v_3 is said to give an *F-triple* for the point p if $p_l \le v_1 < Forw(v_2) < v_2 < Forw(v_3) < v_3 < p_r$ and $v_2 < Forw(v_1) < v_3$ hold. See Fig. 3b for an example. Also, a triple of vertice v_1, v_2 and v_3 is said to give a *B-triple* for the point p if $p_l < v_1 < Backw(v_1) < v_2 < Backw(v_2) < v_3 \le p_r$ and $v_1 < Backw(v_3) < v_2$ hold.

Theorem 2. A simple polygon is not ∞-searchable from the boundary if (**C2**) either the BF-pair or the F-triple occurs for each boundary point, or if (**C3**) either the BF-pair or the B-triple occurs for each boundary point.

Proof. We give below a proof for the condtion **C2**. (The condition **C3** can be proved analogously.) Some examples satisfying **C2** are shown in Fig. 3. Let P be a simple polygon such that **C2** applies, but **C1** doesn't. To show that P is not searchable from the boundary, we distinguish the following three cases.

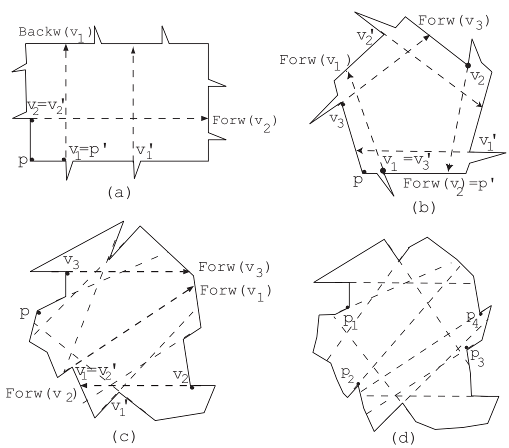

Fig. 3. Several examples satisfying the condition **C2**

Case 1. All boundary points have their BF-pairs (Fig. 3a). For a boundary point p, there are two vertices v_1 and v_2 such that $p_l < v_1 < Backw(v_1) < v_2 < p_r$ and $v_1 < Forw(v_2)$ hold. Suppose that the ∞-searcher starts at p. The ∞-searcher has to move over v_1 or v_2 at least once; otherwise, P cannot be cleared. When the ∞-searcher moves into the interior of the edge $\overline{v_1Succ(v_1)}$ (resp. $\overline{v_2Pred(v_2)}$) at a time, say, t, the target can sneak from $Pred(v_2)$ (resp. $Succ(v_1)$) to the point p, making p be recontaminated. Consider now the point v_1 (resp. v_2) as a new starting point p'. There are also two vertices v_1' and v_2' such that $p_l' < v_1' < Backw(v_1') < v_2' < p_r'$ and $v_1' < Forw(v_2')$ hold. Note that $Pred(v_2')$ and $Succ(v_1')$ are contaminated at the time t. Since all boundary points have their BF-pairs, the starting points considered eventually give a cycle of recontaminations. Hence, P is not searchable from the boundary.

Case 2. All boundary points have their F-triples (Fig. 3b). For a boundary point p, there are two vertices v_2 and v_3 such that $p_l < Forw(v_2) < v_2 < Forw(v_3) < v_3 < p_r$ holds. Assume that v_2 is the maximum vertex satisfying the above inequality, with respect to p. Let p' denote the point $Forw(v_2)$. There are also two vertices v_2' and v_3' such that $p_l' < Forw(v_2') < v_2' < Forw(v_3') < v_3' < p_r'$ holds. Since v_2 is the maximum vertex satisfying $p_l < Forw(v_2) < v_2 < Forw(v_3) < v_3 < p_r$, we have $v_3 \neq v_3'$. Then, $p_l' < Forw(v_3) < Forw(v_3') < v_3 < p_r'$ holds; otherwise, **C1** becomes true due to v_2, v_3 and v_3'. Let us restrict p to be a point of the (half-open) interval $[v_3, v_3')$ and let $v_1 = v_3'$. Then, for any boundary point x, there are three vertices v_1, v_2 and v_3 such that $x_l < p_1 < Forw(p_2) < p_2 < Forw(p_3) < Forw(p_1) < p_3 < x_r$ holds. This inequality gives the F-triple for the point x, and is the key to the following proof.

Suppose that the ∞-searcher starts at a point p. Both $Pred(v_2)$ and $Pred(v_3)$ are contaminated initially. Assume first that the ∞-searcher moves on the boundary of P counterclockwise. We repeatedly consider $Forw(v_2)$ for the current point p as a new starting point p'. When the ∞-searcher moves into the interior of the edge $\overline{v_2Pred(v_2)}$, the target can sneak, say, from $Pred(v_3)$ to $Pred(v_3')$, making p be recontaminated. Since all boundary points have their F-triples, the starting points considered eventually give a cycle of recontaminations. Assume now that the ∞-searcher moves clockwise. When he moves to the interior of the edge $\overline{v_3Pred(v_3)}$, the target can sneak from $Pred(v_2)$ to any point of the interval $[Pred(v_2), v_3)$, making p be recontaminated. Take v_3 as a new starting point. Again, the starting points considered eventually give a cycle of recontaminations. Finally, suppose that the ∞-searcher can change his moving direction. Also, we repeatedly consider $Forw(v_2)$ or v_3 for the current point p as a new starting point, depending on which one is first encountered. As discussed above, a cycle of recontaminations eventually occurs among these starting points. Hence, P is not searchable from the boundary.

Case 3. Some boundary points have the BF-pairs and the others have the F-triples. Let p denote a point, for which the F-triple occurs but the BF-pair does not. Then, there are three vertices v_1, v_2 and v_3 such that $p_l \leq v_1 < Forw(v_2) < v_2 < Forw(v_3) < v_3 < p_r$ and $v_2 < Forw(v_1) < v_3$ hold. Assume that v_1 and v_3 are the maximum and minimum vertices satisfying the above inequalities, with respect to p. Any point $x \in [v_1, v_2] \cup [Forw(v_3), v_3]$ has the BF-triple; otherwise, either all boundary points of P have their F-triples or the condition **C2** cannot be satisfied. Since v_2 and v_3 may contribute to the other F-triple, the boundary of P is divided into two or four intervals such that the BF-pairs and the F-triples appear alternately. In Fig. 3d, each point of the interval $[p_1, p_2]$ or $[p_3, p_4]$ has the F-triple, and each point of the interval $[p_2, p_3]$ or $[p_4, p_1]$ has the BF-pair. In Fig. 3c, two alternate intervals can be found.

From the discussion made in Case 1, it is ineffective for the ∞-searcher to start at a point for which the BF-pair occurs. Consider a search schedule that starts at a point p, for which only the F-triple occurs. Let us see what happens (or which points are contaminated) when the ∞-searcher moves from one interval having the F-triple to the other having the BF-pair. Suppose that the ∞-searcher moves from p to $Forw(v_2)$ counterclockwise in the time interval $[0, t]$, $0 < t$. Let p' denote the point $Forw(v_2)$. Then, there are two vertices v_1', v_2' such that $p_l' < v_1' < Backw(v_1') < v_2' < p_r'$ and $v_1' < Forw(v_2')$ hold (Fig. 3c). Since v_1 is the maximum vertex giving the F-triple for the point p, we have $v_1 = v_2'$. Note that $v_1' < v_2$ holds; otherwise, **C1** becomes true due to v_1', v_2' and v_2. Since $v_1 < Forw(v_2) < v_2 < Forw(v_1)$ holds, the vertex $Succ(v_1')$ is contaminated or recontaminated at the time t.

Let us proceed to show that P is not searchable from the boundary. Assume that all points of the interval $[v_1, v_3]$ have their BF-pairs. At first, when the ∞-searcher moves within the interval $[Pred(v_3), v_1]$, the target can hide himself at $Pred(v_2)$. When the ∞-searcher reaches a point $x \in [v_1, Pred(v_3)]$, the target can always hide himself at the successor of the first vertex of the two giving the BF-pair for x. A cycle of recontaminations occurs when the ∞-searcher moves back to the interval $[Pred(v_3), v_1]$. See Fig. 3c for an example. Assume now that there is a sub-interval of $[v_2, Forw(v_3)]$, whose points have their F-triples (see Fig. 3d). In this case, when the ∞-searcher moves within that sub-interval, the target can hide himself at $Pred(v_3)$. Also, a cycle of recontaminations eventually occurs. It completes the proof. $\square$

3.2 Sufficiency

In this section, we show that the absence of all configurations specified by **C1**, **C2** and **C3** ensures that a polygon is 1-searchable from the boundary.

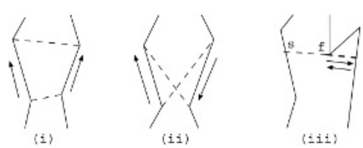

Fig. 4. Instructions for the 1-searcher in boundary search

Consider the elementary actions performed by the 1-searcher [5]. The 1-searcher s and the endpoint f of his flashlight can move along segments of single edges such that (i) no (proper) intersections occur among all segments $\overline{sf}$ during the movement or (ii) any two of the segments $\overline{sf}$ intersect each other, and (iii) f jumps from one point to another point on the boundary of P such that the ray between s and f is extended or shortened. See Fig. 4. The first two

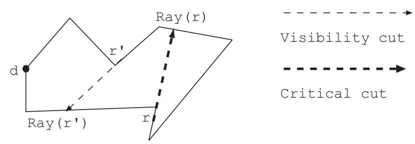

Fig. 5. Visibility cuts and critical cuts

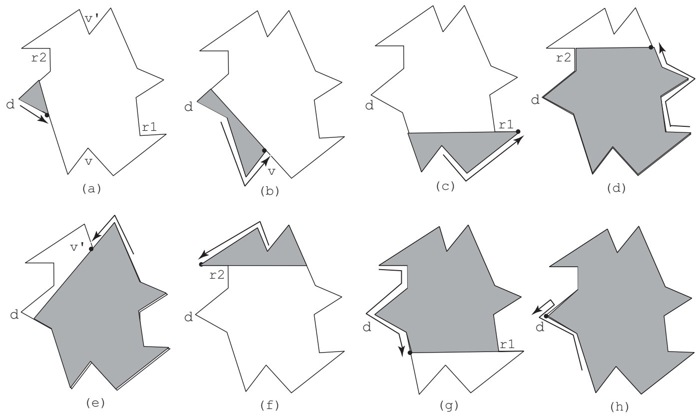

Fig. 6. Snapshots of a search schedule

instructions are allowed for two guards, but the last one is not. So any polygon that is walkable by two guards is 1-searchable from the boundary. We will refer to a *flashlight rotation* as a set of continuous instructions (ii) and (iii), including at least one instruction (ii), and a *walk* as a set of continuous instructions (i) and (ii), including at least one instruction (i).

Let us define the visibility events occurred in any search schedule starting at a boundary point d. Let r denote a reflex vertex. The polygon P can be divided into two pieces by a "cut" that extends an edge incident to r until it hits the boundary of P. A cut is a *visibility cut* if it produces a convex angle at r in the piece of P containing d. Let $Ray(r)$ denote the other endpoint of the visibility cut produced by r, and let $P(\overline{rRay(r)})$ denote the piece of P containing d. A visibility cut $\overline{rRay(r)}$ is *critical* if $P(\overline{rRay(r)})$ is not contained in any other $P(\overline{r'Ray(r')})$, where $\overline{r'Ray(r')}$ is also a visibility cut. See Fig. 5 for an example. We call the reflex vertices, whose visibility/critical cut are defined, the *visibility/critical vertices*.

Our general strategy is to clear the corners incident to critical vertices counterclockwise. But, all these corners as well as the starting point d are allowed to be recontaminated. This is the major difficulty that arises in boundary search. In order to clear the corner incident to a critical vertex, we design a simple "greedy" algorithm, i.e., a walk or a flashlight rotation is performed to the utmost limit. Fig. 6 gives an example, in which two critical vertices r_1 and r_2 are defined.

Snapshots of a search schedule are shown (the arrow shows the movement of the 1-searcher and the cleared region at each step is shaded).

Theorem 3. A simple polygon is 1-searchable from the boundary if none of **C1**, **C2** and **C3** applies.

Proof. Let P be a simple polygon, for which none of **C1**, **C2** and **C3** applies. Then, there is a boundary point d such that none of the BF-pair, the F-triple and the B-triple occurs for d. Order all boundary point of P counterclockwise, starting at d. Both inequalities $d_l < Forw(v_1) < v_1 < Forw(v_2) < v_2 < d_r$ and $d_l < v_1 < Backw(v_1) < v_2 < Backw(v_2) < d_r$ cannot hold simultaneously; otherwise, either the condition **C1** becomes true or the BF-pair for d occurs. In the following, assume that only $d_l < Forw(v_1) < v_1 < Forw(v_2) < v_2 < d_r$ may hold. (The situation in which $d_l < v_1 < Backw(v_1) < v_2 < Backw(v_2) < d_r$ holds can be dealt with analogously.)

Let m denote the number of critical vertices, and let $r_1, \ldots, r_m$ be the sequence of the critical vertices in the increasing order. Denote by $P(r_i)$ and $P - P(r_i)$ the regions which are to the left and right of the segment $\overline{r_i Ray(r_i)}$, respectively. (If d is contained in $P(r_i)$, then $P(r_i) = P(\overline{r_i Ray(r_i)})$; otherwise, $P(r_i) = P - P(\overline{r_i Ray(r_i)})$.) Our search algorithm is so designed that the clear portion of P is always to the left of the ray emanating from the flashlight, as viewed from d. To be exact, we clear the regions $P(r_i)$ in the order $i = 1, \ldots, m$ and finally the whole polygon P (Case 1), except for the situation where $d_l < Forw(v_1) < v_1 < Forw(v_2) < v_2 < d_r$ holds (Case 2).

For a walk or a flashlight rotation, we denote by $R(x1, y1)$ $(x1 \leq y1)$ and $L(x2, y2)$ $(x2 \leq y2)$ the chains on which the 1-searcher s and the endpoint f of his flashlight move, respectively. The ray of the flashlight is often denoted by $\overline{sf}$. A reflex vertex in a chain is called a *blocking* vertex if it blocks one of its adjacent vertices from being visible from any point in the opposite chain.

Case 1. The inequality $d_l < Forw(v_1) < v_1 < Forw(v_2) < v_2 < d_r$ never holds. Let $r_0 = P(r_0) = d$. We will show how to clear the region $P(r_i)$, $i \geq 1$, assuming that $P(r_{i-1})$ has been cleared. The absence of **C1** and the **BF**-pair for d is sufficient for P to be searchable from the boundary in this case.

Case 1.1. $i = 1$. Two subcases are distinguished according to whether d is contained in $P(r_1)$ or not.

Case 1.1.1. The point d is contained in $P(r_1)$. Two chains $R(d_l, r_1)$ and $L(Ray(r_1), d_r)$ are shown by bold lines in Fig. 7. They are mutually weakly visible, and there are no deadlocks between them; otherwise, there are some other critical vertices before r_1 (Fig. 7a-b), the inequality $v_1 < Backw(v_1) < v_2 < Backw(v_2)$ (Fig. 7c) or the BF-pair for d (Fig. 7d-e) holds, or the condition **C1** is true (Fig. 7f), a contradiction. Hence, the region $P(r_1)$ can be cleared by a walk from the point d to the segment $\overline{r_1 Ray(r_1)}$.

Case 1.1.2. The point d is not contained in $P(r_1)$. Assume first that there are no other visibility vertices in the interval $[d_l, r_1]$. Consider the shortest path between d and r_1. Extend all segments of this path until they hit the boundary of P. Let d' denote the other endpoint of the first extended segment (Fig. 8a).

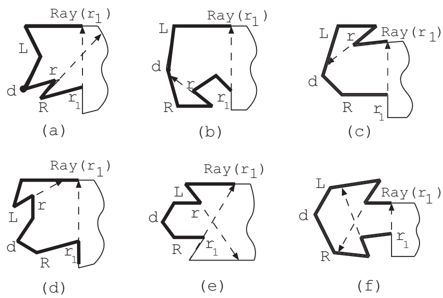

Fig. 7. Case 1.1.1

All points preceding d' are visible from d; otherwise, there are some visibility vertices in $[d_l, r_1]$, a contradiction. The region being to the left of $\overline{d'd}$ (oriented from d' to d) can then be cleared by moving the 1-searcher s from d to d', while keeping f at d (Fig. 9a). If $d' > Ray(r_1)$, then all points of the interval $[d', r_1]$ are visible from the intersection point of $\overline{d'd}$ and $\overline{r_1 Ray(r_1)}$; otherwise, there are some other critical (i.e., blocking) vertices preceding r_1 (and possibly r_1 is not critical), a contradiction. Thus, $P(r_1)$ can be cleared by rotating the flashlight from $\overline{d'd}$ to $\overline{r_1 Ray(r_1)}$. If $d' < Ray(r_1)$, the region $P(r_1)$ can analogously be cleared by rotating the flashlight from $\overline{d'd}$ to $\overline{r_1 Ray(r_1)}$, through the remaining extended segments of the shortest path between d and r_1.

Suppose now that there are some visibility vertices v' before r_1. Clearly, d is contained in these regions $P(\overline{v' Ray(v')})$. Let v denote the maximum of these visibility vertices. Observe that if we ignore (or delete) the critical vertices r whose regions $P(\overline{r Ray(r)})$ contain $P(\overline{v Ray(v)})$, the vertex v as well as some vertices preceding v become *critical.* Then as in Case 1.1.1 and Case 1.2 (see below), we can clear the region $P(\overline{v Ray(v)})$. If $v < Ray(r_1)$, the region $P(r_1)$ can be cleared by finding the shortest path between v and r_1, extending the segments of the path until they hit the boundary of P, and rotating the flashlight through every extended segment of the path (Fig. 8b). Consider the case that $v > Ray(r_1)$. If all points in the chain $R(v, r_1)$ are visible from the intersection point of $\overline{v Ray(v)}$ and $\overline{r_1 Ray(r_1)}$, the flashlight can simply be rotated from $\overline{v Ray(v)}$ to $\overline{r_1 Ray(r_1)}$. Otherwise, let $r^\star$ be the minimum vertex in $R(v, r_1)$ such that $Pred(r^\star)$ is not visible from the intersection point (Fig. 8c). The flashlight can then be rotated from $\overline{v Ray(v)}$ to $\overline{r^\star Forw(r^\star)}$. This procedure is repeatedly performed until the flashlight is rotated to $\overline{r_1 Ray(r_1)}$.

Case 1.2. $1 < i \leq m$. Two subcases are distinguished according to whether $\overline{r_{i-1} Ray(r_{i-1})}$ intersects with $\overline{r_i Ray(r_i)}$ or not.

Case 1.2.1 The segment $\overline{r_{i-1} Ray(r_{i-1})}$ *intersects with* $\overline{r_i Ray(r_i)}$. If the chain $R(r_{i-1}, r_i)$ is weakly visible from $L(Ray(r_{i-1}), Ray(r_i))$, the flashlight is rotated from $\overline{r_{i-1} Ray(r_{i-1})}$ to $\overline{r_i Ray(r_i)}$ as follows. If all points between r_{i-1} and r_i are visible from the intersection point of $\overline{r_{i-1} Ray(r_{i-1})}$ and $\overline{r_i Ray(r_i)}$, the flashlight can be rotated around the intersection point. Otherwise, let $r^\star$ be the minimum

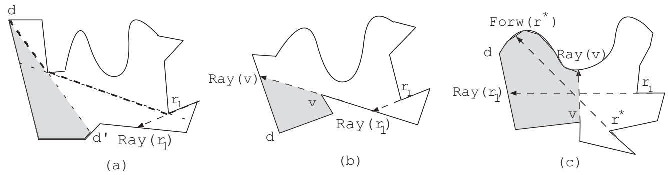

Fig. 8. Case 1.1.2

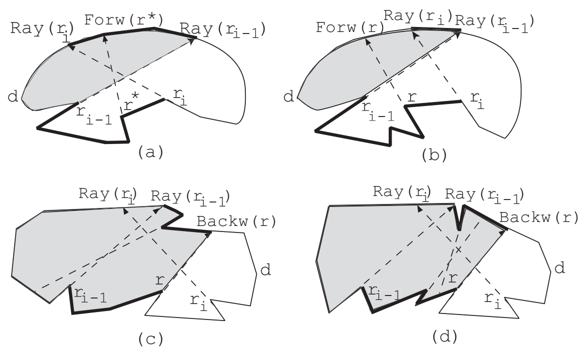

Fig. 9. Case 1.2.1

vertex in $R(r_{i-1}, r_i)$ such that $Pred(r^\star)$ (if d is contained in $P(r_{i-1})$) or $Succ(r^\star)$ (if d is not contained in $P(r_{i-1})$) is not visible from the intersection point. The flashlight can then be rotated from $\overline{r_{i-1}Ray(r_{i-1})}$ to $\overline{r^\star Ray(r^\star)}$. See Fig. 9a for an example. The chain $R(r^\star, r_i)$ is still visible from $L(Ray(r^\star), Ray(r_i))$; otherwise, $R(r_{i-1}, r_i)$ is not weakly visible from $L(Ray(r_{i-1}), Ray(r_i))$ or the blocking vertices in $R(r^\star, r_i)$ are critical, a contradiction. Hence, the flashlight can eventually be rotated to $\overline{r_iRay(r_i)}$.

Suppose now that $R(r_{i-1}, r_i)$ is not weakly visible from $L(Ray(r_{i-1}), Ray(r_i))$. Let r be the blocking vertex in $R(r_{i-1}, r_i)$, whose shot $Ray(r)$ is the furthest from $Ray(r_i)$ on the boundary of P among those of the blocking vertices. See Fig. 9b. The segment $\overline{rRay(r)}$ should exactly intersect with one of $\overline{r_{i-1}Ray(r_{i-1})}$ and $\overline{r_iRay(r_i)}$. The flashlight can first be moved to $\overline{rRay(r)}$, and then to $\overline{r_iRay(r_i)}$; one moverment is a flashlight rotation and the other is a walk. Becasue of our choice of the "furthest" shot $Ray(r)$, the flashlight rotation is always possible. The walk is also possible, since otherwise **C1** becomes true (see Fig. 9c for an example where two considered chains considered are not mutually weakly visible, and Fig. 9d for an example where a deadlock occurs).

Case 1.2.2. The segment $\overline{r_{i-1}Ray(r_{i-1})}$ does not intersect with $\overline{r_iRay(r_i)}$. Following from the definition of critical vertices and the fact that $d_l < Forw(v_1) < v_1 < Forw(v_2) < v_2 < d_r$ does not holds, the point d is contained in $P(r_i)$, but isn't in $P(r_{i-1})$. The chain $R(r_{i-1}, r_i)$ is weakly visible from $L(Ray(r_i), d_r) \cup L(d_l, Ray(r_{i-1}))$; otherwise, there are the critical vertices between r_{i-1} and r_i,

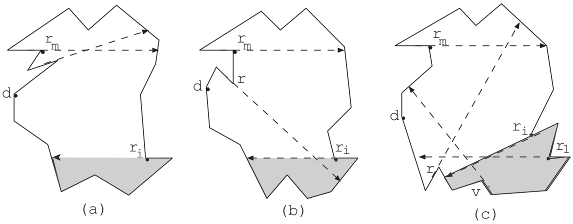

Fig. 10. Case 2.1

or the condition **C1** becomes true due to r_{i-1}, r_i and the blocking vertex in $R(r_{i-1}, r_i)$. The converse is also true; otherwise, **C1** becomes true due to r_{i-1}, r_i and the blocking vertex in $L(Ray(r_i), d_r) \cup L(d_l, Ray(r_{i-1}))$. There are no deadlocks between these two chains; otherwise, **C1** beomes true due to two vertices giving a deadlock and the vertex r_{i-1} or r_i. Hence, the flashlight can be moved from $\overline{r_{i-1}Ray(r_{i-1})}$ to $\overline{r_iRay(r_i)}$ using a walk.

Case 1.3 The region $P(r_m)$ is cleared. If we order all boundary points of P clockwise, then r_m becomes the first critical vertex, and $P - P(r_m)$ is the first region to be cleared. Thus, by a reversed operation of Case 1.1, we can clear the region $P - P(r_m)$ and obtain a complete search schedule. (Note that the 1-searcher traverses the boundary of P only once in Case 1.)

Case 2. The inequality $d_l < Forw(v_1) < v_1 < Forw(v_2) < v_2 < d_r$ holds. Following the definition of critical vertices, this inequality should be satisfied by some pair of critical vertices. The absence of the F-triple for d will be used in this case. (Symmetrically, the absence of the B-triple for d is used in the case that $d_l < v_1 < Backw(v_1) < v_2 < Backw(v_2) < d_r$ holds.)

Case 2.1. There are two consecutive critical vertices r_i and r_{i+1} such that $Forw(r_i) < r_i < Forw(r_{i+1}) < r_{i+1}$ holds. An example of Case 2.1 can be found in Fig. 6, where a complete search schedule is shown. In this case, $Forw(r_h) < Forw(r_i) < r_h < r_i$ holds for $1 \leq h < i$; otherwise, r_h and r_{i+1} give the BF-pair for d (if $Ray(r_h) = Backw(r_h)$), a contradiction. So the segment $\overline{r_hRay(r_h)}$ intersects with $\overline{r_{h+1}Ray(r_{h+1})}$, for $1 \leq h < i$. As in Case 1.1.2 (for $P(r_1)$) and Case 1.2.1, the region $P(r_i)$ can be cleared. The inequality $Forw(r_i) < r_i < Forw(r_{i+1}) < r_{i+1}$ does not affect the operations performed in Case 1.1.2 and Case 1.2.1.

It is difficult to clear the next region $P(r_{i+1})$, as $P(r_i)$ is completely separated from $P(r_{i+1})$. But, we can directly clear the region $P - P(r_m)$ at present time. Note that $Forw(r_{i+1}) < Forw(r_j) < r_{i+1} < r_j$ holds for $i+1 < j \leq m$; otherwise, **C1** becomes true due to r_i, r_{i+1} and r_j. Thus, the segment $\overline{r_iRay(r_i)}$ does not intersect with $\overline{r_mRay(r_m)}$. Two chains $R(r_i, Ray(r_m))$ and $L(r_m, d_r) \cup L(d_l, Ray(r_i))$ are mutually weakly visible; otherwise, there are the critical vertices between r_i and $Ray(r_m) (< r_{i+1})$, preceding $Ray(r_i)$ $(< r_1)$ or succeeding r_m (Fig. 10a), or r_i, r_m and the blocking vertex r in $L(r_m, d_r) \cup L(d_l, Ray(r_i))$ make **C1** (Fig. 10b) be true or give the F-triple for d (Fig. 10c). There are no deadlocks between these two chains; otherwise, **C1** becomes true due to two vertices giving a deadlock and the vertex r_i or r_m. The region $P - P(r_m)$ can then be cleared using a walk from $\overline{r_i Ray(r_i)}$ to $\overline{Ray(r_m) r_m}$. Next, we clear the region $P(r_m)$ by finding the shortest path between r_m and d, extending the segments of the path, and rotating the extended segments intersecting $\overline{r_m Ray(r_m)}$. (Since $P - P(r_m)$ is already cleared, the flashlight has to be rotated only through the extended segments intersecting $\overline{r_m Ray(r_m)}$.) And, move back the flashlight from $\overline{r_m Ray(r_m)}$ to $\overline{Ray(r_i) r_i}$ using a walk so that the region $P - P(r_i)$ is cleared. Note that the 1-searcher s is now located at $Ray(r_i)$. It is important to see that no instructions (iii) are used in the work of clearing the region $P(r_h)$ (Case 1.1.2 and Case 1.2.1), $1 \leq h \leq i$; otherwise, there is a vertex r ($< Ray(r_h)$) such that r, r_h (e.g., r_i in Fig. 10c) and r_m give the F-triple for d, a contradiction. So the operation done for clearing $P(r_i)$ can *reversely* be performed, even in the sense that the roles of the 1-searcher s and the endpoint f of the flashlight are exchanged. It completes the search schedule for clearing P.

Case 2.2. No two of consecutive critical vertices r and r' satisfy the inequality $Forw(r) < r < Forw(r') < r'$. See Fig. 11 for some examples. Without loss of generality, assume that there are two critical vertices r_i and r_j satisfying the inequality $Forw(r_i) < r_i < Forw(r_j) < r_j$, $i + 1 < j$. As discussed in Case 2.1, $Forw(r_h) < Forw(r_i) < r_h < r_i$ holds for $1 \leq h < i$, and $Forw(r_j) < Forw(r_l) < r_j < r_l$ holds for $j < l \leq m$. Then, as in Cases 1.1.2 and 1.2.1, the region $P(r_m)$ can be cleared. Assume that r_k is the maximum among the critical vertices r satisfying $Forw(r) < r < Forw(r_m) < r_m$. So we have $Ray(r_{k+1}) = Backw(r_{k+1})$ and $r_{k+1} > Forw(r_m)$. By an argument similar to that made in Case 2.1, we can show that the flashlight can be moved from $\overline{r_m Ray(r_m)}$ to $\overline{Ray(r_k) r_k}$ using a walk. This clears the region $P - P(r_k)$. Finally, as shown in Case 2.1, the operation of clearing $P(r_k)$ can reversely be performed. It completes the search schedule for clearing P.

All cases above ensure that P is 1-searchable from the boundary. □

Theorem 4. Any polygon that is ∞-searchable from the boundary is also 1-searchable from the boundary.

Proof. It immediately follows from Theorems 1, 2 and 3. □

4 Algorithm and Complexity

In this section, we give the algorithms for testing the searchability of simple polygons, and reporting a search schedule if it exists.

Theorem 5. It takes $O(n \log n)$ time and $O(n)$ space to determine the searchability of simple polygons.

Proof. Let P be a simple polygon. All ray shots can be computed in $O(n \log n)$ time [1]. In the following, we first present a procedure for finding the vertices of P for which the BF-pairs occur, and then extend it to find the vertices for

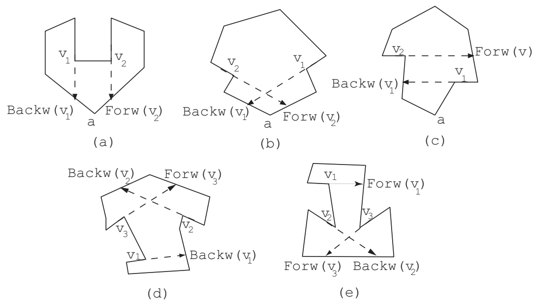

Fig. 11. Illustration of the proof of Theorem 5

which the F-triples (resp. B-triples) occur. Whether or not the condition **C2** (resp. **C3**) is true for P can then be determined from these computed results. Also, the condition **C1** can similarly be verified.

Let a denote an arbitrary vertex of P. Order all vertices and ray shots counterclockwise, starting at a. Let v_1 denote the minimum vertex such that $v_1 < Backw(v_1)$ holds, and v_2 the maximum vertex such that $Forw(v_2) < v_2$ holds. If only one of v_1 and v_2 is found, no BF-pairs occur for the polygon P and we are done. If $v_1 > v_2$ holds, the BF-pairs occur only for the vertices between v_1 and v_2 (Fig. 11a), and we are done. If $Forw(v_2) < v_1 < v_2 < Backw(v_1)$ holds, neither v_1 nor v_2 can contribute to a BF-pair for a (Fig. 11b). In this case, we search for the vertex v_1' next to v_1 such that $v_1' < Backw(v_1')$ holds and the vertex v_2' next to v_2 such that $Forw(v_2) < v_2$ holds, and then call the same procedure to test if v_1' and v_2' give a BF-pair for a. If $v_1 < Forw(v_2) < v_2 < Backw(v_1)$ holds, the region $P(\overline{v_1 Backw(v_1)})$ (containing the point a) is contained in $P(\overline{v_2 Forw(v_2)})$. See Fig. 11c for an example. (The situation in which $Forw(v_2) < v_1 < Backw(v_1) < v_2$ holds can be dealt with analogously.) Then, v_1 cannot contribute to any BF-pair for a. We search for the vertex v_1' next to v_1 such that $v_1' < Backw(v_1')$ holds, and call the testing procedure with the new pair of v_1' and v_2. If $v_1 < Backw(v_1) < v_2$ and $v_1 < Forw(v_2) < v_2$ hold, a BF-pair occurs for all vertices of $[v_2, a_r] \cup [a_l, v_1]$. Next, take $Succ(v_1)$ as the new starting point a', and order all critical vertices and their shots. Since it is equivalent to take a_l' and a_r' as the minimum and maximum points respectively, this ordering can be obtained in constant time. Then search for the minimum vertex v_1' such that $v_1' < Backw(v_1')$ holds, and call the same procedure to test if v_1' and v_2 give the BF-pair for a'. This procedure is terminated when the BF-pair for a is verified again. Clearly, the time taken to find the vertices of P having the BF-pairs is $O(n)$.

In order to compute the vertices of P having the F-triple, we find, for each vertex a, two vertices v_0, v_1 such that v_1 is the minimum vertex satisfying $v_0 < Forw(v_1) < v_1 < Forw(v_0)$, and two vertices v_1', v_2 such that v_1' is the maximum vertex satisfying $Forw(v_1') < v_1' < Forw(v_2) < v_2$. If $v_1 \leq v_1'$, then v_0, v_1 and v_2 give an F-triple for a; otherwise, no F-triples occur for a. By an argument similar to that for computing the vertices of P having the B-triples, we can find, in $O(n)$ time, the vertices of P for which the F-triples (resp. B) occur. We leave the detail to readers.

Turn to the condition **C1**. Using Das et al.'s algorithm [2], we can find in linear time if there are three *critical vertices* (all boundary points are considered as the starting point once) such that no intersections occur among three segments connecting a critical vertex with its ray shot. If *yes*, the condition **C1** is true. The remaining situations in which **C1** applies are shown in Fig. 11d-e. By an argument similar to that for finding an F-triple or a B-triple, we can verify in $O(n)$ time if the situation shown in Fig. 11d or Fig. 11e occurs or not.

Finally, the space requirement of our algorithm is $O(n)$. □

Theorem 6. A search schedule can be reported in time linear in its size I, if it exists. For the ∞-searcher, $I < 2n$, and for the 1-searcher, $I < 3n^2$.

Proof. Let P be a simple polygon, for which none of **C1**, **C2** and **C3** applies. Then, there is a boundary point d such that at most one of $Forw(v_1) < v_1 < Forw(v_2) < v_2$ and $v_1 < Backw(v_1) < v_2 < Backw(v_2)$ holds, and no BF-pairs occur for d. To obtain a search schedule for the 1-searcher, we run the constructive algorithm presented in the proof of Theorem 3. Clearing a region $P(r_i)$ consists of a number of flashlight rotations and walks. If we consider the polygonal chain, which is traversed by the 1-searcher for the second or third time, as a new different chain, the chains $R(x, y)$ considered for all walks and for all flashlight rotations as well are disjoint. Since the total size of these chains is equal to n in Case 1, and smaller than $3n$ in Case 2, the number I of search instructions output is smaller than $3n^2$ (see also [4]).

Consider now the size of search schedules for the ∞-searcher. We directly apply the search algorithm given in the proof of Theorem 3 to the ∞-searcher. In Case 1, a complete search schedule is obtained before or when the ∞-searcher returns to d. In Case 2.1, we claim that a complete search schedule is obtained when or before the ∞-searcher moves to the point $Ray(r_i)$ for the second time. Consider the walk from $\overline{r_m Ray(r_m)}$ to $\overline{Ray(r_i) r_i}$, which is performed after $P(r_m)$ is cleared. The movement of the ∞-searcher along the chain $R(r_m, d_r) \cup R(d_l, Ray(r_i))$ clears the chain $L(r_i, Ray(r_m))$ in Case 2.1. Now, we show that the remaining chain from $Ray(r_i)$ to r_i, denoted by $L'(Ray(r_i), r_i)$, is also cleared by this movement of the ∞-searcher. Assume that there are no visibility vertices in the interval $[r_m, d_r]$; otherwise, the flashlight can be moved to the maximum of these visibility vertices and its (forward) ray shot, using a walk. Then, two chains $L'(Ray(r_i), r_i)$ and $R(r_m, d_r) \cup R(d_l, Ray(r_i))$ are mutually weakly visible; otherwise, r_m and the blocking vertex in $L'(Ray(r_i), r_i)$ give the BF-pair for the point d, or r_i, r_m and the blocking vertex in $R(d_l, Ray(r_i))$ give the F-triple for d, a contradiction. Thus, any point of $L'(Ray(r_i), r_i)$ has to be illuminated once during the movement of the ∞-searcher from r_m to $Ray(r_i)$, and any clear point can never be recontaminated. Our claim is proved. Also, in Case 2.2, a complete search schedule is obtained when or before the ∞-searcher moves to the point $Ray(r_k)$ for the second time. Hence, we have $I < 2n$. □

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Δ-Optimum Exclusive Sum Labeling of Certain Graphs with Radius One

Mauritsius Tuga[1] and Mirka Miller[2]

[1] School of Electrical Engineering and Computer Science, The University of Newcastle, NSW 2308 Australia
moris@cs.newcastle.edu.au

[2] School of Information Technology and Mathematical Science, University of Ballarat, VIC 3353 Australia
m.miller@ballarat.edu.au

Abstract. A mapping L is called a sum labeling of a graph $H(V(H), E(H))$ if it is an injection from $V(H)$ to a set of positive integers, such that $xy \in E(H)$ if and only if there exists a vertex $w \in V(H)$ such that $L(w) = L(x) + L(y)$. In this case, w is called a *working vertex.* We define L as *an exclusive sum labeling* of a graph G if it is a sum labeling of $G \cup \overline{K_r}$ for some non negative integer r, and G contains no working vertex. In general, a graph G will require some isolated vertices to be labeled exclusively. The least possible number of such isolated vertices is called *exclusive sum number* of G; denoted by $\epsilon(G)$.

An exclusive sum labeling of a graph G is said to be *optimum* if it labels G exclusively by using $\epsilon(G)$ isolated vertices. In case $\epsilon(G) = \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of vertices in G, the labeling is called Δ*-optimum exclusive sum labeling.*

In this paper we present Δ-optimum exclusive sum labeling of certain graphs with radius one, that is, graphs which can be obtained by joining all vertices of an integral sum graph to another vertex. This class of graphs contains infinetely many graphs including some populer graphs such as wheels, fans, friendship graphs, generalised friendship graphs and multicone graphs.

1 Introduction

All graphs we consider in this paper are finite,simple, undirected graphs. For a graph $G(V(G), E(G))$ of order n and size m, let $\Delta(G)$ denote the maximum degree of its vertices. A labeling $L : V(G) \to (Z)Z^+$ of G is called *(integral) sum labeling* if for any two distinct vertices u and v , $uv \in \mathrm{E}(G)$ if and only if there exists a vertex $w \in V(G)$ with $L(w) = L(u) + L(v)$. In such a case the vertex w is said to be a *working vertex.* It is obvious that if G has a sum labeling then it has at least one isolated vertex. If G has a (integral) sum labeling then G is called a *(integral) sum graph.* The notion of a sum graph was introduced by Harary in 1990 [3]. In 1994 Harary introduced the notion of an integral sum graph [4].

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In general, a graph G requires some r ($r \geq 0$) isolated vertices to be a (integral) sum graph. The smallest such r is called the *(integral) sum number* of G; it is denoted by $(\zeta(G))\sigma(G)$.

A sum labeling L is said to be *exclusive* with respect to a graph G if it labels $G \cup \overline{K_r}$ for some non negative integer r in such a way that G contains no working vertex. In this case we say that L is an exclusive sum labeling of G. Otherwise, L is said to be an *inclusive* sum labeling of G. The notion of an exclusive sum labeling is introduced in [6]. Any given graph G will require some isolated vertices to be labeled exclusively. The least possible number of isolated vertices that need to be added to a graph G to be labeled exclusively is called the *exclusive sum number* of the graph G, denoted by $\epsilon(G)$. Obviously, $\epsilon(G)$ must be at least equal to the maximum vertex degree $\Delta(G)$. It is interesting to investigate graphs which have a small exclusive sum number. We introduce the following.

Definition 1. A sum labeling of graph $G \cup \overline{K_{\Delta(G)}}$ such that G contains no working vertex is said to be Δ-*optimum* exclusive sum labeling of G.

In this paper we present constructions of Δ-optimum exclusive sum labelings of certain graphs with radius one.

2 Graphs with Radius One

The class of graphs with radius one can be described as containing graphs that can be obtained by joining all vertices of a given graph G to a vertex o not in G. We let $G + o$ denote a new graph obtained by connecting all vertices in G to an isolated vertex o not in G.

It is clear that $G + o$ is a connected graph with $V(G + o) = V(G) \cup \{o\}$ and $E(G + o) = E(G) \cup \{op \mid p \in V(G)\}$.

Recall that

1. A *fan* f_n can be obtained by joining all vertices of a path (P_n) to another vertex o, that is, $f_n = P_n + o$.
2. A *friendship* graph F_m can be obtained by joining all vertices of m copies of matchings (K_2) to another vertex o, that is, $F_m = mK_2 + o$.
3. A *wheel* W_n can be obtained by joining all vertices of a cycle C_n to a further vertex o, that is, $W_n = C_n + o$.

The abovementioned graphs are members of the class of graphs with radius one. Additionally, this class also contains many other graphs, including *generalised friendship graph* and *multicone graph.*

Definition 2. *A generalised friendship graph F_m^n is a graph obtained by joining all vertices of m copies of a path P_n to a further vertex o, that is, $F_m^n = mP_n + o$.*

Definition 3. *A multicone graph M_m^n is a graph obtained by joining all vertices of m copies of a cycle C_n to a further vertex o, that is, $M_m^n = mC_n + o$.*

The following theorem is the main result of this paper.

Theorem 1. *Let G be an ISG (Integral Sum Graph) with n vertices which has an integral sum labeling φ such that $\varphi(x) \neq 0$ for every vertex $x \in V(G)$. Then $\epsilon(G+o) = \Delta(G+o) = n$.*

Proof

Let G be an integral sum graph with n vertices $\{P_i \mid i = 1, 2, ..., n\}$. Let L be a non zero integral sum labeling of G where $L(P_i) = a_i \neq 0, \quad \forall i = 1, 2, ..., n$.

Let $\overline{K_n}$ be a graph of n isolated vertices with $V(\overline{K_n}) = \{T_i \mid i = 1, 2, ..., n\}$.

Let $H = (G+o) \cup \overline{K_n}$, then $V(H) = V(G) \cup \{o\} \cup V(\overline{K_n})$.

Choose an integer c such that $c \geq 3(max\{\mid a_i \mid, \ i = 1, 2, ..., n\}$ and $3 \not| c$.

Label the vertices of H with $L^{'}$ in the following way,

$$\begin{aligned} L^{'}(o) &= c \\ L^{'}(P_i) &= b_i = 3a_i + c \\ L^{'}(T_i) &= t_i = b_i + c = 3a_i + 2c \end{aligned}$$

Let

$$\begin{aligned} &S = \{c, b_1, b_2, ..., b_n, t_1, t_2, ..., t_n\} \\ &B = \{b_1, b_2, ..., b_n\} \\ &T = \{t_1, t_2, ..., t_n\} \end{aligned}$$

We will show that $L^{'}$ is a sum labeling of H. Moreover, it is an exclusive sum labeling of $G+o$.

1. By the following facts, $L^{'}$ is a labeling of $V(H)$ with distinct possitive numbers (injective) .
   (i) Since $a_i \neq a_j$ if $i \neq j$, then $b_i \neq b_j$ and $t_i \neq t_j$, for $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$.
   (ii) c is not divisible by 3, hence $b_i \neq t_j$ for all $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$.
   (iii) Since $a_i \neq 0 \quad \forall i = 1, 2, ..., n$, then $b_i \neq c \quad \forall i = 1, 2, ..., n$.
   (iv) Using the fact that $c > 3 \mid a_i \mid$, we have $t_i \neq c \quad \forall i = 1, 2, ..., n$.
2. **(i)** If $P_iP_j \in E(G)$ then

$$\begin{aligned} L^{'}(P_i) + L^{'}(P_j) &= L^{'}(P_i) + L^{'}(P_j) \\ &= 3(a_i) + c + 3(a_j) + c \\ &= 3(a_i + a_j) + 2c \\ &= (3a_k + c) + c \\ &= b_k + c = t_k \in T \subset S. \end{aligned}$$

   (ii) For $i = 1, 2, ..., n, L^{'}(o) + L^{'}(P_i) = c + b_i = t_i \in T \subset S$.
   Hence if $xy \in E(H)$ then $L^{'}(x) + L^{'}(y) \in S$.

3. We will show that no unwanted edge is induced by the labeling, that is, $xy \notin E(H) \Rightarrow L^{'}(x) + L^{'}(y) \notin S$.
   For convenience, we shall use the label of each vertex of H under $L^{'}$ to denote the vertex itself. We consider the following cases.

   Case 1. Let $x = b_i, y = b_j$ and $(b_i, b_j) \notin$ E(H). We will show that $b_i + b_j \notin S$. Suppose on the contrary that

   (i) If $b_i + b_j = b_k$ for some k, then

$$\begin{aligned} 3a_i + c + 3a_j + c &= 3a_k + c \\ 3a_i + 3a_j + c &= 3a_k \\ a_i + a_j + c/3 &= a_k \end{aligned}$$

   This is a contradiction to the fact that $3 \not| \, c$, and $a_i \in \mathbb{Z}, \ \forall i = 1, 2, ...n$.

   (ii) If $b_i + b_j = c$ then $a_i + a_j + c/3 = 0$. Again, this is impossible since the $a_i^{'}s$ are integers and $3 \not| \, c$.

   (iii) If $b_i + b_j = t_k$ for some k, then

$$\begin{aligned} b_i + b_j &= b_k + c \\ 3a_i + c + 3a_j + c &= 3a_k + 2c \\ a_i + a_j &= a_k \end{aligned}$$

   We get $P_iP_j \in E(G)$ and therefore $b_ib_j \in E(H)$. This contradicts the fact that $b_ib_j \notin E(H)$.

   We leave the details to the reader to check the following cases.

   Case 2. Let x $= b_i, y = t_j$. Then it is easy to prove
   $b_i + t_j \neq b_k, \quad \forall k$
   $b_i + t_j \neq c$
   $b_i + t_j \neq t_k, \quad \forall k.$

   Case 3. Let $x = t_i, y = t_j$. Then it is easy to prove
   $t_i + t_j \neq b_k, \quad \forall k$
   $t_i + t_j \neq c$
   $t_i + t_j \neq t_k, \quad \forall k.$

   Case 4. Let x = c , y= t_j. Then it is easy to prove
   $c + t_j \neq b_k, \quad \forall k$
   $c + t_j \neq c$
   $c + t_j \neq t_k, \quad \forall k.$

By the above three facts, it is clear that $L^{'}$ is a sum labeling of H. Moreover, because $G + o$ contains no working vertex, we conclude that $L^{'}$ is an exclusive sum labeling of $G + o$ using n isolated vertices. Therefore, $\epsilon(G + o) \leq n$. On the other hand, $\Delta(G + o) = n$, which gives $\epsilon(G + o) \geq n$. We conclude that $\epsilon(G + o) =| V(G) |= \Delta(G + o) = n$. □

The following results are useful for further application of the theorem.

– *[S1] (*Harary *et al. [3]) For all positive integers n, the path P_n is an integral sum graph.*
  Note that for $n \leq 3$ the optimal labeling contains the label 0.
– *[S2] (*Harary *et al. [3]) For all positive integers m, the matching mK_2 is an integral sum graph.*
  However, if $m = 1$, the optimal labeling contains 0.
– *[S3] (*Wu *et al. [5]) If G is an integral sum graph without $0 - vertex$, then mG is also an integral sum graph for each $m \in \mathbb{N}$.*
– *[S4] (*Sharary *[8]) The integral sum number of cycles is given by:*

$$\zeta(C_n) = \begin{cases} 3 & \text{when} \quad n = 4 \\ 0 & \text{when} \quad n \neq 4 \end{cases}$$

– *[S5] (*Xu *et al.) For an arbitrary integer $m \geq 1$, mK_3 is an integral sum graph.*

3 Exclusive Sum Labeling of Particular Classes of Graphs

In this section we give exclusive sum labelings of particular classes of graphs with radius one. The following is a corrolary of [S1]-[S5] and Theorem 1.

Corollary 1. 1. $\epsilon(f_n) = n, \quad n \geq 4$
2. $\epsilon(F_m) = 2m, \quad m \geq 2$
3. $\epsilon(F_m^n) = mn, \quad m \geq 2, n \geq 4$
4. $\epsilon(W_n) = n, \quad n \geq 5$
5. $\epsilon(f_m^n) = mn, \quad m \geq 2, n \geq 3$

Proof

1. Let $G = P_n, \quad n \geq 4$. Note that $G + o \cong f_n$ and $\Delta(P_n + o) = n$. Applying Theorem 1 and [S1], we have $\epsilon(f_n) = n$, for $n \geq 4$. □

   For example, let $n = 7$. The following labeling is an integral sum labeling of P_7.

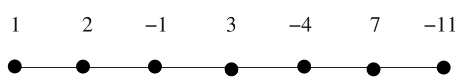

Fig. 1. Integral sum labeling of P_7

Take $c = 34$. Construct P_7+o $\cong f_7$, and label all the vertices in the same way as in the proof of Theorem 1. We obtain:

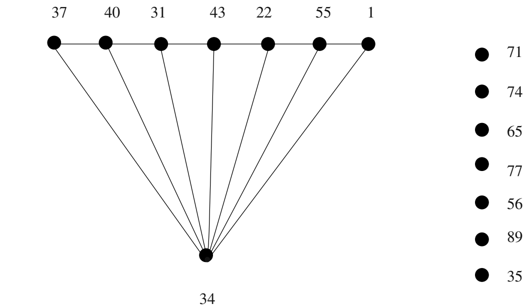

Fig. 2. Exclusive labeling of fan f_7

2. Let $m \geq 2$ and $G = mK_2$. It is clear that $G + o \cong F_m$ and $\Delta(G + o) = 2m$. Therefore, using Theorem 1 and [S3], we have $\epsilon(F_m) = 2m$. □

For example, let $m = 4$. Figure 3. shows an integral sum labeling of $4K_2$. Take $c = 61$. Construct $4K_2 + o \cong F_4$ and label all the vertices in the same way as in the proof of Theorem 1. We obtain an exclusive sum labeling of the graph F_4 as depicted in Figure 4.

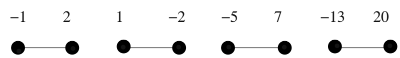

Fig. 3. Integral sum labeling of $4K_2$

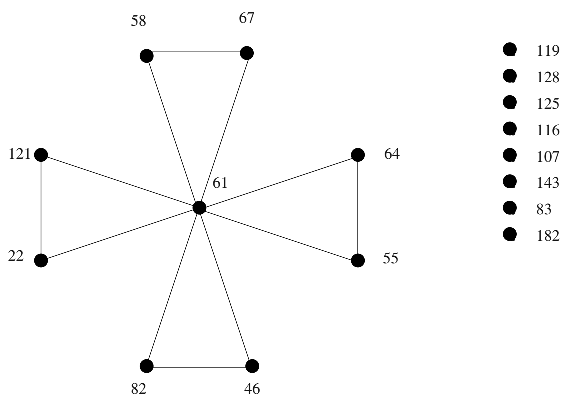

Fig. 4. Exclusive sum labeling of friendship graph F_4

3. Let $m \geq 2, n \geq 4$ and $G = mP_n$. It is clear that $G + o \cong F_m^n$ and $\Delta(G + o) = mn$. Using Theorem 1,[S2], [S4] we get $\epsilon(F_m^n) = mn$. □

For example, let $m = 4, n = 4$. Figure 5. presents an integral sum labeling of $4P_4$. Take $c = 1945$. Construct $4P_4 + o \cong F_4^4$ and label all the vertices

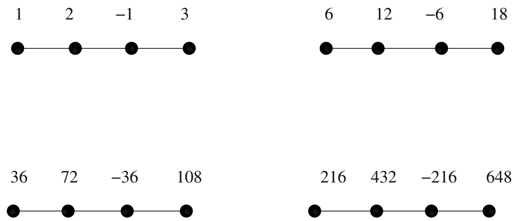

Fig. 5. Integral sum labeling of $4P_4$

in the same way as in the proof of Theorem 1. We obtain an exclusive sum labeling of F_4^4 as shown in Figure 6.

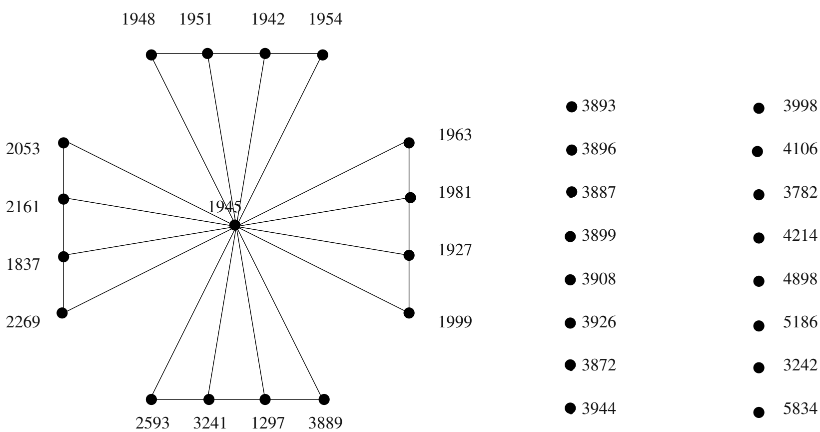

Fig. 6. Exclusive sum labeling of general friendship graph F_4^4

4. Let $n \geq 5$ and $G = C_n$. It is clear $G + o \cong W_n$ and $\Delta(G + o) = n$. Therefore by using Theorem 1 and [S5], we have $\epsilon(W_n) = n$. $\square$

For example, let $n = 7$. Figure 7. shows an integral sum labeling of C_7. Take $c = 22$. Construct $C_7{+}o \cong W_7$ and label all the vertices in the same way

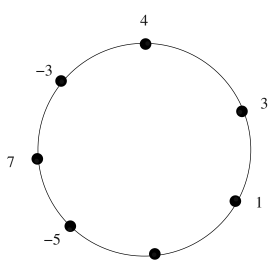

Fig. 7. Integral sum labeling of C_7

as in the proof of Theorem 1. We obtain an excluisve sum labeling of W_7 as shown in Figure 8.

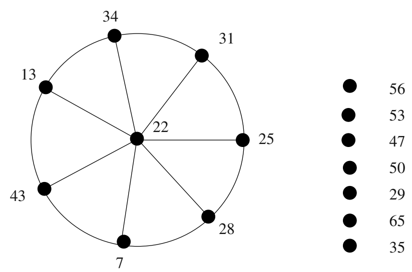

Fig. 8. Exclusive sum labeling of wheel W_7

5. Let $m \geq 2, n \geq 3$ and $G = mC_n$. We notice that $G+o \cong M_m^n$ and $\Delta(G+o) = mn$. Therefore by using Theorem 1, [S2] and [S4] we have $\epsilon(M_m^n) = mn$. □

   For example, let $m = 4, n = 3$. Figure 9. presents an integral sum labeling of $4C_3$. Take $c = 316$. Construct $4C_3$+o $\cong M_4^3$ and label all the vertices in the

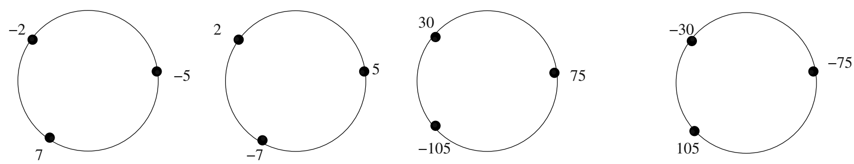

Fig. 9. Integral sum labeling of $4C_3$

   same way as in the proof of Theorem 1. We obtain an exclusive sum labeling of M_4^3 as shown in Figure 10.

Let $S \subset \mathbb{N}$ be a finite set of positive integers. A sum graph of S, denoted by $G^+(S)$, is a graph with vertex set S where two vertices are adjacent if and only if their sum is in S. A bill graph B_n is a sum graph of S where $S = \{1, 2, \ldots, n\}$. Obviously, for every $n \geq 3, \sigma(B_n) = 0$, and if we connect every vertex of B_n to another isolated vertex, then we obtain a new graph H with $H \cup \overline{K_1} \cong B_{n+2}$.

Note that for every n, B_n is an integral sum graph which does not contain 0 as its label. Therefore, Theorem 1 can be applied to bill graph and we have the following.

Corollary 2. *For $n > 3, \epsilon(B_n) = \Delta(B_n) = n - 2$.*

Example. Consider the bill graph B_6 in Figure 11.

Take $c = 13$ and relabel the connected part with $b_i = 13 + 3(i - 1), i = 1, 2, ., n - 1$, relabel the old isolated vertex with 80 and label $n - 2$ new isolated vertices with $t_i = 36 + 3i, i = 1, 2, 3, 4$. We get an exclusive sum labeling of the graph, as shown in Figure 12.

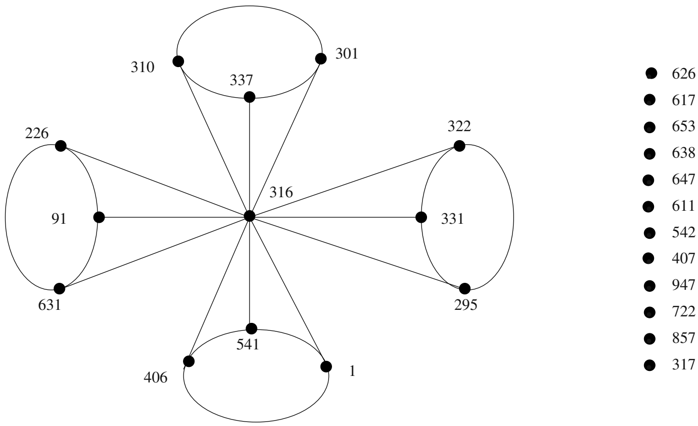

Fig. 10. Exclusive sum labeling of multicone (M_4^3)

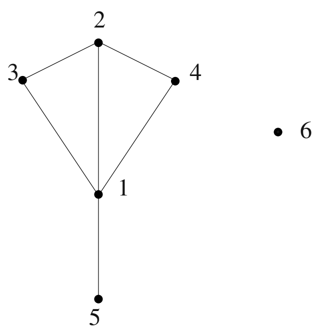

Fig. 11. Sum labeling of bill graph B_6

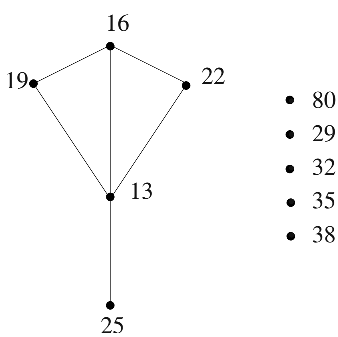

Fig. 12. Exclusive sum labeling of bill graph B_6

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Author Index